

Fibre Products of Supersingular Curves and the Enumeration of Irreducible Polynomials with Prescribed Coefficients

Omran Ahmadi¹, Faruk Göloğlu², Robert Granger^{*3†}, Gary McGuire^{4‡}, and Emrah Sercan Yilmaz^{4‡}

¹ School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
oahmadid@ipm.ir

² Department of Algebra, Charles University Prague
farukgolloglu@gmail.com

³ School of Computer and Communication Sciences, École polytechnique fédérale de Lausanne, Switzerland
robert.granger@epfl.ch

⁴ School of Mathematics and Statistics, University College Dublin, Ireland
gary.mcguire@ucd.ie, emrahsercanyilmaz@gmail.com

Abstract. For any positive integers $n \geq 3, r \geq 1$ we present formulae for the number of irreducible polynomials of degree n over the finite field \mathbb{F}_{2^r} where the coefficients of x^{n-1} , x^{n-2} and x^{n-3} are zero. Our proofs involve counting the number of points on certain algebraic curves over finite fields, a technique which arose from Fourier-analysing the known formulae for the \mathbb{F}_2 base field cases, reverse-engineering an economical new proof and then extending it. This approach gives rise to fibre products of supersingular curves and makes explicit why the formulae have period 24 in n .

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1 Introduction

For p a prime and $r \geq 1$ let \mathbb{F}_q denote the finite field of $q = p^r$ elements. The number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n is usually denoted $I_q(n)$, and the formula for $I_q(n)$ is a classical result due to Gauss [6, pp. 602-629]. A natural extension problem is to determine the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree n for which certain coefficients are prescribed. An interesting set of subproblems here consists of counting the number of monic irreducible polynomials of degree n for which the first l coefficients have the prescribed values t_1, \dots, t_l , while the remaining coefficients are arbitrary, i.e., considering polynomials of the form

$$x^n + t_1x^{n-1} + \dots + t_lx^{n-l} + a_{l+1}x^{n-l-1} + \dots + a_{n-1}x + a_n,$$

whose number we denote by $I_q(n, t_1, \dots, t_l)$. While the asymptotics for such subproblems have been obtained by Cohen [3], several exact results are known. Carlitz gave formulae for $I_q(n, t_1)$ in 1952 [1], and Kuz'min gave formulae for $I_q(n, t_1, t_2)$ in 1990 [11, 12]. For the base field \mathbb{F}_2 , Cattell *et al.* reproduced Kuz'min's results in 1999 [2], while in 2001 Yucas and Mullen computed $I_2(n, t_1, t_2, t_3)$ for n even [19] and Fitzgerald and Yucas computed $I_2(n, t_1, t_2, t_3)$ for n odd [5]. In 2013 Ri *et al.* gave formulae for $I_{2^r}(n, t_1, t_2)$ for all $r \geq 1$ [16]. In this paper we give a formula for $I_{2^r}(n, 0, 0, 0)$ for all $r \geq 1$.

While our aim is to count the number of monic irreducible polynomials with certain coefficients prescribed, it turns out to be easier (and arguably more interesting) to count the number of elements of the relevant fields with correspondingly prescribed traces; a standard Möbius inversion-type argument allows one to switch between the two. Indeed, [2, 19, 5, 16] all take this approach. With this in mind,

*Corresponding author.

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for $r \geq 1$ let $q = 2^r$. For $a \in \mathbb{F}_{q^n}$, the characteristic polynomial of a w.r.t. the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is defined to be:

$$\prod_{i=0}^{n-1} (x + a^{q^i}) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0. \quad (1)$$

The coefficient a_{n-1} is known as the trace of a , while a_{n-2} is known as the subtrace. We are concerned with the first three traces, which we henceforth denote by T_1, T_2 and T_3 . By (1) they are given by the following expressions:

$$\begin{aligned} T_1(a) &= \sum_{i=0}^{n-1} a^{q^i}, \\ T_2(a) &= \sum_{0 \leq i < j \leq n-1} a^{q^i + q^j}, \\ T_3(a) &= \sum_{0 \leq i < j < k \leq n-1} a^{q^i + q^j + q^k}. \end{aligned}$$

For $t_1 \in \mathbb{F}_q$, let $F_q(n, t_1)$ be the number of elements $a \in \mathbb{F}_{q^n}$ for which $T_1(a) = t_1$. Similarly, for $t_1, t_2 \in \mathbb{F}_q$ let $F_q(n, t_1, t_2)$ be the number of elements $a \in \mathbb{F}_{q^n}$ for which $T_1(a) = t_1$ and $T_2(a) = t_2$. Lastly, for $t_1, t_2, t_3 \in \mathbb{F}_q$ let $F_q(n, t_1, t_2, t_3)$ be the number of elements $a \in \mathbb{F}_{q^n}$ for which $T_1(a) = t_1, T_2(a) = t_2$ and $T_3(a) = t_3$.

The exact counts of elements with prescribed traces found in [2, 19, 5, 16] are intriguing since they depend on $n \bmod 8$ in the two trace ($l = 2$) cases and on $n \bmod 24$ in the three trace ($l = 3$) cases. In this work we give a new explanation for this phenomenon. It happens because the formulae are derived from the number of rational points on certain curves – and the important point is that these curves are supersingular. The supersingularity implies that the roots of the Weil polynomials are roots of unity. The presence of these roots of unity explains the periodicity.

The algebraic curves that arise in this paper have a certain form, namely $S(y) = T(x)$ where $T(x) = xR_1(x) + (xR_2(x))^2 + \cdots + (xR_m(x))^{2^{m-1}}$ and each S, R_i is a 2-linearised polynomial. Such curves have been studied by van der Geer-van der Vlugt [7]. Following them, we show that our curves can be viewed as fibre products of simpler curves, and we compute the trace of Frobenius for the simpler curves.

Finding exact formulae for $F_q(n, t_1, \dots, t_l)$ when $l > 3$ is an open and apparently much harder problem. We note that Koma has given approximations for these counts in the case $q = 2$ and $l = 4$, when $n \equiv 0, 2 \pmod{4}$ [10]. In forthcoming work by the third author, by using a similar curve-based approach it is shown that the formulae for when more than three coefficients are prescribed are in general not periodic, as they do not arise from supersingular curves alone. This fact explains why previous approaches to this problem – which did not use our curve-based approach – have failed to make any progress in the past 15 years. Thus, in the present work we explain the periodicity of known formulae, extend this to larger base fields, and lay the foundations for further developments in this area.

The paper is organised as follows. In Section 2 we present the formula for $I_q(n, 0, 0, 0)$ in terms of $F_q(n, 0, 0, 0)$, then in Section 3 we provide some background on curves and their Jacobians. In Section 4 we motivate our approach by considering the \mathbb{F}_2 base field case with the first two traces specified. We then extend this and consider the same base field but with the first three traces specified, in Section 5. In Sections 6, 7 and 8, we compute $F_q(n, 0, 0, 0)$ for all finite binary fields, and make some concluding remarks in Section 9.

2 Computing $I_q(n, 0, 0, 0)$ from $F_q(n, 0, 0, 0)$

In this section we express $I_q(n, 0, 0, 0)$ in terms of $F_q(n, 0, 0, 0)$, using simple extensions of results from [2] and [19]. Recall that $I_q(n)$ is the number of irreducible polynomials of degree n in $\mathbb{F}_q[x]$ and $I_q(n, t_1)$ is the number of irreducible polynomials of degree n in $\mathbb{F}_q[x]$ with first coefficient t_1 . Furthermore, let $\mu(\cdot)$ be the Möbius function, and for a proposition P let $[P]$ denote its truth value, i.e., $[P] = 1$ if P is true and 0 if P is false, with 1 and 0 interpreted as integers. The formula is given by the following theorem.

Theorem 1. *Let $n \geq 3$. Then*

$$I_q(n, 0, 0, 0) = \frac{1}{n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) (F_q(n/d, 0, 0, 0) - [n \text{ even}] q^{n/2d-1}) \quad (2)$$

$$= \frac{1}{n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) F_q(n/d, 0, 0, 0) - [n \text{ even}] I_{\sqrt{q}}(n, 1). \quad (3)$$

For $\beta \in \mathbb{F}_{q^n}$ let $p = \text{Min}(\beta)$ denote the minimum polynomial of β over \mathbb{F}_q , which has degree n/d for some $d \mid n$. Note that $T_i(\beta)$ is the coefficient of x^{n-i} in p^d [2, Lemma 2], and so abusing notation slightly we also write $T_i(\beta)$ as $T_i(p^d)$. We use the following easy lemma, which was stated for the $q = 2$ case in [19, Prop. 1].

Lemma 1. *For each integer $d \geq 1$ and $p(x) \in \mathbb{F}_q[x]$,*

1. $T_1(p^d) = dT_1(p)$
2. $T_2(p^d) = \binom{d}{2}T_1(p) + dT_2(p)$
3. $T_3(p^d) = \binom{d}{3}T_1(p) + dT_3(p)$

Note that $\binom{d}{i} \bmod 2$ for $i = 1, 2, 3$ depends on $d \bmod 4$. In particular, $\binom{d}{2}$ is even if and only if $d \equiv 0, 1 \pmod{4}$ and $\binom{d}{3}$ is even if and only if $d \equiv 0, 1, 2 \pmod{4}$. Throughout we therefore write $d \equiv a$ to denote $d \equiv a \pmod{4}$. As in [2] let $\mathbf{Irr}(n)$ denote the set of all irreducible polynomials of degree n over \mathbb{F}_q , and let $a \cdot \mathbf{Irr}(n)$ denote the multiset consisting of a copies of $\mathbf{Irr}(n)$. As in [2] and [19] we have:

$$\begin{aligned} F_q(n, t_1, t_2, t_3) &= \left| \bigcup_{\beta \in \mathbb{F}_{q^n}, T_1(\beta)=t_1, T_2(\beta)=t_2, T_3(\beta)=t_3} \text{Min}(\beta) \right| \\ &= \left| \bigcup_{d|n} \frac{n}{d} \left\{ p \in \mathbf{Irr}\left(\frac{n}{d}\right) : dT_1(p) = t_1, \binom{d}{2}T_1(p) + dT_2(p) = t_2, \binom{d}{3}T_1(p) + dT_3(p) = t_3 \right\} \right| \\ &= \left| \bigcup_{d|n, d \equiv 0} \frac{n}{d} \left\{ p \in \mathbf{Irr}\left(\frac{n}{d}\right) : 0 = t_1, 0 = t_2, 0 = t_3 \right\} \right| \\ &\quad + \left| \bigcup_{d|n, d \equiv 1} \frac{n}{d} \left\{ p \in \mathbf{Irr}\left(\frac{n}{d}\right) : T_1(p) = t_1, T_2(p) = t_2, T_3(p) = t_3 \right\} \right| \\ &\quad + \left| \bigcup_{d|n, d \equiv 2} \frac{n}{d} \left\{ p \in \mathbf{Irr}\left(\frac{n}{d}\right) : 0 = t_1, T_1(p) = t_2, 0 = t_3 \right\} \right| \\ &\quad + \left| \bigcup_{d|n, d \equiv 3} \frac{n}{d} \left\{ p \in \mathbf{Irr}\left(\frac{n}{d}\right) : T_1(p) = t_1, T_1(p) + T_2(p) = t_2, T_1(p) + T_3(p) = t_3 \right\} \right|. \end{aligned}$$

For $t_1 = t_2 = t_3 = 0$ we have

$$F_q(n, 0, 0, 0) = \sum_{\substack{d|n \\ d \equiv 0}} \frac{n}{d} I_q\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 0\right) + \sum_{\substack{d|n \\ d \text{ odd}}} \frac{n}{d} I_q\left(\frac{n}{d}, 0, 0, 0\right) \quad (4)$$

To evaluate the sum of the first two terms of (4) we employ the following analogue of [2, Lemma 5].

Lemma 2.

$$\sum_{\substack{d|n \\ d \equiv 0}} \frac{n}{d} I_q\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 0\right) = [n \text{ even}] q^{n/2-1}.$$

Proof. Recall the classical result of Gauss [6, pp. 602-629], and that of Carlitz [1], for $t_1 \neq 0$, respectively:

$$I_q(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}, \quad I_q(n, t_1) = \frac{1}{qn} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) q^{n/d}, \quad (5)$$

and consequently $I_q(n, 0) = I_q(n) - (q-1)I_q(n, 1)$. Let

$$A(n) = (q-1) \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 1\right)$$

and let

$$B(n) = \sum_{\substack{d|n \\ d \equiv 0}} \frac{n}{d} I_q\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 0\right).$$

If n is odd then the sums in $B(n)$ are empty and hence the Lemma is true. Therefore let n be even. By essentially the same argument given in the proof of [2, Lemma 5], we have $A(n) = (q-1)q^{n/2-1}$. Similarly, we have

$$\begin{aligned} B(n) + A(n) &= \sum_{\substack{d|n \\ d \equiv 0}} \frac{n}{d} I_q\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 0\right) + (q-1) \sum_{\substack{d|n \\ d \equiv 2}} \frac{n}{d} I_q\left(\frac{n}{d}, 1\right) \\ &= \sum_{\substack{d|n \\ d \text{ even}}} \frac{n}{d} I_q\left(\frac{n}{d}\right) = \sum_{d|\frac{n}{2}} \frac{n}{2d} I_q\left(\frac{n}{2d}\right) = q^{n/2}. \end{aligned}$$

Hence $B(n) = q^{n/2} - A(n) = q^{n/2} - (q-1)q^{n/2-1} = q^{n/2-1}$. □

Using this result, (4) becomes

$$F_q(n, 0, 0, 0) - [n \text{ even}] q^{n/2-1} = \sum_{\substack{d|n \\ d \text{ odd}}} \frac{n}{d} I_q\left(\frac{n}{d}, 0, 0, 0\right).$$

Applying [2, Corollary 1] with $\alpha(n) = nI_q(n, 0, 0, 0)$ finally proves expression (2) of Theorem 1, and applying Carlitz's result from (5) gives (3).

3 Background on curves and Jacobians

Let p be a prime number and let $q = p^r$. Let C be a curve of genus g defined over \mathbb{F}_q . All curves in this paper are projective plane curves, lying in $\mathbb{P}^2(k)$ where k is an algebraic closure of \mathbb{F}_q . As is common, we will work with affine versions of our curves and include the point at infinity in our point counts. Note that we do not include the point at infinity when counting the relevant $F_q(n, t_1, t_2, t_3)$.

We denote by J_C the Jacobian of C , an abelian variety which as a group is isomorphic to the divisor class group $\text{Pic}^0(C)$ (the degree 0 divisors modulo the principal divisors).

The map $(x, y) \mapsto (x^q, y^q)$ on C (defined over an algebraic closure of \mathbb{F}_q) induces an endomorphism on J_C called the Frobenius endomorphism. The characteristic polynomial of the Frobenius endomorphism plays a key role. The roots of this characteristic polynomial are called Frobenius eigenvalues.

3.1 Abelian varieties

Suppose that A and B are Abelian varieties over the same field K . A homomorphism $f : A \rightarrow B$ is a morphism that is also a group homomorphism. A homomorphism is an isogeny over K if f is surjective and defined over K and $\dim(A) = \dim(B)$. If A is an Abelian variety over K , A is called simple (over K) if it is not isogenous over K to a product of lower-dimensional Abelian varieties.

Because J_C is an abelian variety we continue with some facts from the theory of abelian varieties.

Let A be any abelian variety of dimension g over \mathbb{F}_q . It can be shown that A has a Frobenius endomorphism, and that the characteristic polynomial $P_A(t) \in \mathbb{Z}[t]$ of the Frobenius endomorphism has the form

$$P_A(t) = t^{2g} + a_1 t^{2g-1} + \dots + a_{g-1} t^{g+1} + a_g t^g + q a_{g-1} t^{g-1} + \dots + a_1 q^{g-1} t + q^g.$$

The Weil polynomial of an abelian variety over a finite field \mathbb{F}_q is the characteristic polynomial of its Frobenius endomorphism. The Weil polynomial of a curve over \mathbb{F}_q is the Weil polynomial of its Jacobian.

The isogeny classes of abelian varieties are completely classified by their Weil polynomials, as the following theorem of Tate shows.

Theorem 2. (Tate) *Let A and B be abelian varieties defined over \mathbb{F}_q . Then an abelian variety A is \mathbb{F}_q -isogenous to an abelian subvariety of B if and only if $P_A(t)$ divides $P_B(t)$ over $\mathbb{Q}[t]$. In particular, $P_A(t) = P_B(t)$ if and only if A and B are \mathbb{F}_q -isogenous.*

3.2 L-polynomials

For a curve C over \mathbb{F}_q we define for any $n \geq 1$ the number $N_n(C)$ of \mathbb{F}_{q^n} rational points of C , and we encode these numbers into the zeta function of C

$$Z_C(t) = \exp\left(\sum_{n \geq 1} N_n(C) \frac{t^n}{n}\right).$$

It can be shown that the zeta function is a rational function of t with denominator $(1-t)(1-qt)$. The numerator of the zeta function of a curve C is called the L-polynomial of C . It can be shown that the L-polynomial of C is the reverse polynomial of the Weil polynomial of J_C , which we denote $P_C(t)$, i.e.,

$$L_C(t) = t^{2g} P_C(1/t).$$

Let $P_C(t) = \prod_{i=1}^{2g} (t - \eta_i)$, so $L_C(t) = \prod_{i=1}^{2g} (1 - \eta_i t)$. It can be shown that if $L_n(t)$ denotes the L-polynomial of C over the extension \mathbb{F}_{q^n} , then $L_n(t) = \prod_{i=1}^{2g} (1 - \eta_i^n t)$. We also have that $N_n := |C(\mathbb{F}_{q^n})| = q^n + 1 - \sum_{i=1}^{2g} \eta_i^n$ for all $n \geq 1$, and the Riemann Hypothesis $|\eta_i| = \sqrt{q}$, which together yield $|N_n - (q^n + 1)| \leq 2g\sqrt{q^n}$ (the Hasse-Weil bound).

In this paper, all L-polynomials are for curves defined over \mathbb{F}_2 unless otherwise stated.

3.3 Supersingularity

An elliptic curve E over \mathbb{F}_q is called supersingular if its Weil polynomial $t^2 + at + q$ has the property that a is divisible by p . Equivalently, E is supersingular if $E(\overline{\mathbb{F}_q})$ has no points of order p . A curve C of genus g over a finite field \mathbb{F}_q is called supersingular if its Jacobian is a supersingular abelian variety, i.e., its Jacobian is isogenous (over $\overline{\mathbb{F}_q}$) to E^g , where E is a supersingular elliptic curve.

For a supersingular curve C defined over a finite field \mathbb{F}_q , it was shown by Oort that $\eta_i = \sqrt{q} \cdot \zeta_i$ where ζ_i is a (not necessarily i -th) root of unity, for all i .

3.4 Maximal and minimal curves

Let g be genus of the curve C defined over \mathbb{F}_{2^r} and let η_1, \dots, η_{2g} be the roots of reciprocal of the L -polynomial of C over \mathbb{F}_{2^r} . Then the number of rational points of C on $\mathbb{F}_{2^{rn}}$ is

$$\#C(\mathbb{F}_{2^{rn}}) = (2^{rn} + 1) - \sum_{i=1}^{2g} \eta_i^n.$$

As stated earlier $|\eta_i| = \sqrt{2^r}$ for all $i = 1, \dots, 2g$. We call $C(\mathbb{F}_{2^{rn}})$ maximal if $\eta_i^n = -\sqrt{2^{rn}}$ for all $i = 1, \dots, 2g$, so the Hasse-Weil upper bound is met. We call $C(\mathbb{F}_{2^{rn}})$ minimal if $\eta_i^n = \sqrt{2^{rn}}$ for all $i = 1, \dots, 2g$.

A supersingular curve defined over \mathbb{F}_q becomes maximal over some finite extension of \mathbb{F}_q , because $\eta_i = \sqrt{q} \cdot \zeta_i$ where ζ_i is a (not necessarily i -th) root of unity, for all i .

Proposition 1. *If $C(\mathbb{F}_{2^{rn}})$ is maximal or minimal for some $n \geq 1$ then $C(\mathbb{F}_{2^{r(2n)}})$ is minimal.*

Proof. Since $\eta_i^n = \pm\sqrt{2^{rn}}$ for all $i = 1, \dots, 2g$, we have $\eta_i^{2n} = (\pm\sqrt{2^{rn}})^2 = \sqrt{2^{r(2n)}}$ for all $i = 1, \dots, 2g$. The proof follows by definition of a minimal curve. \square

We will need the following simple observations later.

Lemma 3. *The real numbers 1 and $\sqrt{2}$ cannot be written as a \mathbb{Q} -linear combination of primitive 16-th roots of unity. Moreover, 1 cannot be written as a \mathbb{Q} -linear combination of primitive 8-th roots of unity.*

Proof. Let $\eta_{2^n} = \exp(\pi/2^{n-1}) \in \mathbb{C}$ be a primitive 2^n -th root of unity. Then η_{2^n} is a root of $X^{2^{n-1}} + 1$ which is the 2^n -th cyclotomic polynomial and therefore irreducible over \mathbb{Q} . Let $K_n := \mathbb{Q}(\eta_{2^n})$. Then $1, \eta_{2^n}, \dots, \eta_{2^n}^{2^{n-1}-1}$ are linearly independent over \mathbb{Q} . Note that $\eta_{2^n}^i$ is a primitive 2^n -th root of unity if and only if i is odd.

Since $1 = 1$ and $\sqrt{2} = \eta_{16}^2 - \eta_{16}^6$, 1 and $\sqrt{2}$ cannot be written as \mathbb{Q} -linear combination of primitive 16-th roots of unity. Since $1 = 1$, 1 cannot be written as \mathbb{Q} -linear combination of primitive 8-th roots of unity. \square

3.5 Quadratic forms

We now recall the basic theory of quadratic forms over \mathbb{F}_2 , which we will use later.

Let $K = \mathbb{F}_{2^m}$, and let $Q : K \rightarrow \mathbb{F}_2$ be a quadratic form. The polarization of Q is the symplectic bilinear form B defined by

$$B(x, y) = Q(x + y) - Q(x) - Q(y).$$

By definition the radical of B (denoted W) is

$$W = \{x \in K : B(x, y) = 0 \text{ for all } y \in K\}.$$

The rank of B is defined to be $m - \dim(W)$, and a well known theorem states that the rank must be even.

Next let $Q|_W$ denote the restriction of Q to W , and let

$$W_0 = \{x \in W : Q(x) = 0\}$$

(sometimes W_0 is called the singular radical of Q). Note that $Q|_W$ is a linear map $W \rightarrow \mathbb{F}_2$ with kernel W_0 . Therefore

$$\dim W_0 = \begin{cases} \dim(W) - 1 & \text{if } Q|_W \text{ is onto} \\ \dim(W) & \text{if } Q|_W = 0 \text{ (i.e. } W = W_0\text{).} \end{cases}$$

Table 1. $f_2(n, t_1, t_2)$

$n \pmod{8}$	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
0	$-2^{n/2-1}$	$2^{n/2-1}$	0	0
1	$2^{n/2-3/2}$	$-2^{n/2-3/2}$	$2^{n/2-3/2}$	$-2^{n/2-3/2}$
2	0	0	$-2^{n/2-1}$	$2^{n/2-1}$
3	$-2^{n/2-3/2}$	$2^{n/2-3/2}$	$2^{n/2-3/2}$	$-2^{n/2-3/2}$
4	$2^{n/2-1}$	$-2^{n/2-1}$	0	0
5	$-2^{n/2-3/2}$	$2^{n/2-3/2}$	$-2^{n/2-3/2}$	$2^{n/2-3/2}$
6	0	0	$2^{n/2-1}$	$-2^{n/2-1}$
7	$2^{n/2-3/2}$	$-2^{n/2-3/2}$	$-2^{n/2-3/2}$	$2^{n/2-3/2}$

The rank of Q is defined to be $m - \dim(W_0)$. The following theorem is well known, see Chapter 6 of [13] for example.

Proposition 2. *Continue the above notation. Let $N = |\{x \in K : Q(x) = 0\}|$, and let $w = \dim(W)$.*

If Q has odd rank then $N = 2^{m-1}$. In this case, $\sum_{x \in K} (-1)^{Q(x)} = 0$.

If Q has even rank then $N = 2^{m-1} \pm 2^{(m-2+w)/2}$.

4 Motivating example: $F_2(n, t_1, t_2)$

Since precisely half of the elements of \mathbb{F}_{2^n} have trace zero and half of the elements have trace one, we have $F_2(n, 0) = F_2(n, 1) = 2^{n-1}$ for all $n \geq 1$. Hence the four $F_2(n, t_1, t_2)$ are the first interesting cases.

Cattell *et al.* have expressed $F_2(n, t_1, t_2)$ as sums of binomial coefficients [2, Thm. 4]. For $n \geq 2$, we instead write $F_2(n, t_1, t_2) = 2^{n-2} + f_2(n, t_1, t_2)$ and deduce the formulae for $f_2(n, t_1, t_2)$ from [5, Thm. 3.4] for odd n , and [19, Thm. 3] for even n , which are presented in Table 1, in which the rightmost four columns cover the four (t_1, t_2) pairs. As one can see, for each (t_1, t_2) the expression for $f_2(n, t_1, t_2)$ depends on $n \pmod{8}$. This observation naturally gives rise to two related questions: why are the formulae periodic, and why is the period 8?

To answer these questions we Fourier-analyse the formulae, using the complex 8-th roots of unity. In particular, observe that dividing each $f_2(n, t_1, t_2)$ by $2^{n/2}$ gives a constant. We can thus normalise by this factor and expand each as follows:

$$f_2(n, t_1, t_2) = \sum_{k=0}^7 g(t_1, t_2)_k (\sqrt{2}\omega_8^k)^n = 2^{n/2} \sum_{k=0}^7 g(t_1, t_2)_k \omega_8^{kn}, \quad (6)$$

where the $g(t_1, t_2)_k$ are unknown constants and $\omega_8 = e^{i\pi/4} = (1+i)/\sqrt{2}$. For each (t_1, t_2) , let \bar{v}_{t_1, t_2} be the length-8 vector $f_2(n, t_1, t_2)/2^{n/2}$, i.e.,

$$\begin{aligned} \bar{v}_{0,0} &= (-2^{-1}, 2^{-3/2}, 0, -2^{-3/2}, 2^{-1}, -2^{-3/2}, 0, 2^{-3/2}), \\ \bar{v}_{0,1} &= (2^{-1}, -2^{-3/2}, 0, 2^{-3/2}, -2^{-1}, 2^{-3/2}, 0, -2^{-3/2}), \\ \bar{v}_{1,0} &= (0, 2^{-3/2}, -2^{-1}, 2^{-3/2}, 0, -2^{-3/2}, 2^{-1}, -2^{-3/2}), \\ \bar{v}_{1,1} &= (0, -2^{-3/2}, 2^{-1}, -2^{-3/2}, 0, 2^{-3/2}, -2^{-1}, 2^{-3/2}). \end{aligned}$$

Let M be the discrete Fourier transform matrix for ω_8 with entries $M_{j,k} = \omega_8^{(j-1) \cdot (k-1)}$. Then $\bar{g}(t_1, t_2) = [g(t_1, t_2)_0, \dots, g(t_1, t_2)_7]$ satisfies:

$$M \cdot \bar{g}(t_1, t_2)^T = \bar{v}_{t_1, t_2}^T.$$

Since $M_{j,k}^{-1} = \frac{1}{8} \cdot \omega_8^{-(j-1) \cdot (k-1)}$, we easily compute

$$\bar{g}(t_1, t_2)^T = M^{-1} \cdot \bar{v}_{t_1, t_2}^T.$$

Then substituting each $\bar{g}(t_1, t_2)$ into the Fourier expansion (6), we obtain

Proposition 3. *For $n \geq 2$ we have*

$$\begin{aligned} f_2(n, 0, 0) &= -\frac{1}{4}((\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) = -\frac{1}{4}((-1+i)^n + (-1-i)^n), \\ f_2(n, 0, 1) &= \frac{1}{4}((\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) = \frac{1}{4}((-1+i)^n + (-1-i)^n), \\ f_2(n, 1, 0) &= -\frac{i}{4}((\sqrt{2}\omega_8^3)^n - (\sqrt{2}\omega_8^5)^n) = -\frac{i}{4}((-1+i)^n - (-1-i)^n), \\ f_2(n, 1, 1) &= \frac{i}{4}((\sqrt{2}\omega_8^3)^n - (\sqrt{2}\omega_8^5)^n) = \frac{i}{4}((-1+i)^n - (-1-i)^n). \end{aligned}$$

By a theorem due to Gölöğlu, McGuire and Moloney, the expressions for $F_2(n, 0, 0)$, $F_2(n, 1, 1)$, $F_2(n, 0, 1)$ and $F_2(n, 1, 0)$ count the number of elements $a \in \mathbb{F}_{2^n}$ for which the corresponding binary Kloosterman sum is congruent to 0, 4, 8 and 12 (mod 16), respectively [8, Thm. 1.2]. Furthermore, Lisoněk and Moiso have explicitly computed the number of elements $a \in \mathbb{F}_{2^n}$ for which the Kloosterman sum is divisible by 16 [14, Thm 3.6], relating it to the number of \mathbb{F}_{2^n} -rational points on the supersingular elliptic curve

$$E_1/\mathbb{F}_2 : y^2 + y = x^3 + x.$$

By Gölöğlu *et al.*'s result, this number is therefore related to $F_2(n, 0, 0)$.

We will make repeated use of the following easy lemma, which is an immediate generalisation of [5, Lemma 1.1], which was proven for the base field \mathbb{F}_2 case.

Lemma 4. *For all $\alpha, \beta \in \mathbb{F}_{q^n}$ we have:*

- (i) $T_2(\alpha + \beta) + T_2(\alpha) + T_2(\beta) = T_1(\alpha)T_1(\beta) + T_1(\alpha\beta)$
- (ii) $T_3(\alpha + \beta) + T_3(\alpha) + T_3(\beta) = T_2(\alpha)T_1(\beta) + T_1(\alpha)T_2(\beta) + T_1(\alpha^2\beta + \alpha\beta^2) + T_1(\alpha\beta)T_1(\alpha + \beta).$

We can now reprove the connection between $F_2(n, 0, 0)$ and $\#E_1(\mathbb{F}_{2^n})$ directly as follows. If $T_1(a) = 0$ then $a = x^2 + x$ for two $x \in \mathbb{F}_{2^n}$. By Lemma 4(i), we have $T_2(x^2 + x) = T_1(x^3 + x)$, and so for $n \geq 2$ we have:

$$\begin{aligned} F_2(n, 0, 0) &= \frac{1}{2} \#\{x \in \mathbb{F}_{2^n} \mid T_1(x^3 + x) = 0\} \\ &= \frac{1}{4} \#\{(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \mid y^2 + y = x^3 + x\} \\ &= \frac{1}{4} (\#E_1(\mathbb{F}_{2^n}) - 1) \\ &= 2^{n-2} - \frac{(-1+i)^n + (-1-i)^n}{4}, \end{aligned}$$

where in the final line we have used the fact that the characteristic polynomial of Frobenius of E_1 is $P_{E_1}(X) = X^2 + 2X + 2 = (X - \sqrt{2}\omega_8^3)(X - \sqrt{2}\omega_8^5)$, since the trace of E_1 is -2 . As $F_2(n, 0, 0) + F_2(n, 0, 1) = F_2(n, 0) = 2^{n-1}$, we immediately have $f_2(n, 0, 1) = -f_2(n, 0, 0)$. The periodicity for the trace zero cases thus arises from the connection with the supersingular elliptic curve E_1 , whose Weil q -numbers are $\sqrt{2}\omega_8^{\pm 3}$. For the trace one cases, note that the form of $f_2(n, 1, 0)$ and $f_2(n, 1, 1)$ implies that they do not arise from the zeta function of an abelian variety, but are related to $f_2(n, 0, 0)$ and $f_2(n, 0, 1)$, as derived in Lemma 2.6 of [5] for odd n , and [19] (deduced from [2]) for even n . Since our focus is on the application of supersingular curves to the counting problem, we omit these cases for now, but indicate how supersingular curves may also be used for the trace one cases, for odd n , in the next section.

Table 2. $f_2(n, 0, t_2, t_3)$

$n \pmod{24}$	$(0, 0, 0)$	$(0, 0, 1)$	$(0, 1, 0)$	$(0, 1, 1)$
0	$-5 \cdot 2^{n/2-2}$	$3 \cdot 2^{n/2-2}$	$2^{n/2-2}$	$2^{n/2-2}$
1	$3 \cdot 2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$
2	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
3	0	$-2^{n/2-3/2}$	$-2^{n/2-3/2}$	$2^{n/2-1/2}$
4	0	$2^{n/2-1}$	0	$-2^{n/2-1}$
5	$-3 \cdot 2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$
6	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
7	$3 \cdot 2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$
8	$-2^{n/2-1}$	0	$-2^{n/2-1}$	$2^{n/2}$
9	0	$2^{n/2-3/2}$	$2^{n/2-3/2}$	$-2^{n/2-1/2}$
10	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
11	$-3 \cdot 2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$
12	$3 \cdot 2^{n/2-2}$	$-2^{n/2-2}$	$-3 \cdot 2^{n/2-2}$	$2^{n/2-2}$
13	$-3 \cdot 2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$
14	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
15	0	$2^{n/2-3/2}$	$2^{n/2-3/2}$	$-2^{n/2-1/2}$
16	$-2^{n/2-1}$	0	$-2^{n/2-1}$	$2^{n/2}$
17	$3 \cdot 2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$
18	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
19	$-3 \cdot 2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$	$2^{n/2-5/2}$
20	0	$2^{n/2-1}$	0	$-2^{n/2-1}$
21	0	$-2^{n/2-3/2}$	$-2^{n/2-3/2}$	$2^{n/2-1/2}$
22	$2^{n/2-2}$	$-2^{n/2-2}$	$2^{n/2-2}$	$-2^{n/2-2}$
23	$3 \cdot 2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$	$-2^{n/2-5/2}$

5 Recomputing $F_2(n, 0, t_2, t_3)$

We now apply the approach of the previous section to compute $F_2(n, 0, t_2, t_3)$. For the general case where t_1 may be 0 or 1, there are two complementary results already known. Fitzgerald and Yucas gave formulae for $F_2(n, t_1, t_2, t_3)$ when n is odd [5], while Yucas and Mullen gave formulae for $F_2(n, t_1, t_2, t_3)$ when n is even [19]. As in Section 4 we Fourier-analyse these formulae in order to determine which supersingular curves are relevant and then prove the formulae directly.

As before, for $n \geq 3$ we write $F_2(n, 0, t_2, t_3) = 2^{n-3} + f_2(n, 0, t_2, t_3)$. Table 2 gives the set of formulae for $f_2(n, 0, t_1, t_2)$, in which the rightmost four columns cover the possible $(0, t_2, t_3)$ triples. As in Section 4, since the set of formulae have period 24, one can express each in terms of the complex 24-th roots of unity. We thus expand each as follows:

$$f_2(n, 0, t_2, t_3) = \sum_{k=0}^{23} g(0, t_2, t_3)_k (\sqrt{2}\omega_{24}^k)^n = 2^{n/2} \sum_{k=0}^{23} g_k \omega_{24}^{kn}, \quad (7)$$

with $\omega_{24} = e^{i\pi/12} = ((1+\sqrt{3}) + (-1+\sqrt{3})i)/2\sqrt{2}$. For each (t_2, t_3) , let \bar{v}_{0,t_2,t_3} be the vector of constants $f_2(n, 0, t_2, t_3)/2^{n/2}$. Let M be the discrete Fourier transform matrix with $M_{j,k} = \omega_{24}^{(j-1) \cdot (k-1)}$. Then $\bar{g}(0, t_2, t_3) = [g(0, t_2, t_3)_k]_{0 \leq k \leq 23}$ satisfies:

$$M \cdot \bar{g}(0, t_2, t_3)^T = \bar{v}_{0,t_2,t_3}^T.$$

Since $M_{j,k}^{-1} = \frac{1}{24}\omega_{24}^{-(j-1) \cdot (k-1)}$ we easily compute

$$\bar{g}(0, t_2, t_3)^T = M^{-1} \cdot \bar{v}_{0,t_2,t_3}^T.$$

Then substituting each $\bar{g}(t_1, t_2)$ into the Fourier expansion (7), we obtain

Proposition 4. For $n \geq 3$ we have

$$\begin{aligned}
f_2(n, 0, 0, 0) &= -\frac{1}{4}((\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) - \frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) \\
&\quad - \frac{1}{8}((\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 0, 0, 1) &= \frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) \\
&\quad + \frac{1}{8}((\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 0, 1, 0) &= -\frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) \\
&\quad + \frac{1}{8}((\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 0, 1, 1) &= \frac{1}{4}((\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) + \frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) \\
&\quad - \frac{1}{8}((\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n).
\end{aligned}$$

As before the $(\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n$ term arises from the roots of the characteristic polynomial of Frobenius $P_{E_1}(X) = X^2 + 2X + 2$. The term $(\sqrt{2}i)^n + (-\sqrt{2}i)^n$ arises from the roots of $P_{E_2}(X) = X^2 + 2$ corresponding to the elliptic curve

$$E_2/\mathbb{F}_2 : y^2 + y = x^3 + 1.$$

Finally the term $(\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n$ arises from the roots of $P_{H_1}(X) = X^4 + 2X^3 + 2X^2 + 4X + 4$ corresponding to the genus 2 supersingular curve (see [18, §4])

$$H_1/\mathbb{F}_2 : y^2 + y = x^5 + x^3.$$

In order to prove Proposition 4 starting from the relevant supersingular curves, we use the following easy lemma due to Fitzgerald. For a function $q_i : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ let $Z(q_i)$ denote the number of zeros of q_i , and for $w_i \in \mathbb{F}_2$ let $N(w_1, w_2)$ be the number of $x \in \mathbb{F}_{2^n}$ such that $q_i(x) = w_i$, for $i = 1, 2$.

Lemma 5. [4, Prop. 5.3] Let $q_1, q_2 : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ be functions. Then

$$N(0, 0) = \frac{1}{2}(Z(q_1) + Z(q_2) + Z(q_1 + q_2) - 2^n).$$

Note that Fitzgerald stipulates that q_1, q_2 be quadratic forms, since his application requires only this; the proof however only requires that they be functions. We prove a generalisation of Lemma 5 in Section 6. We now prove Proposition 4.

Proof. By Lemma 4(i) we have $T_2(x^2 + x) = T_1(x^3 + x)$ and by Lemma 4(ii), we have $T_3(x^2 + x) = T_1(x^5 + x)$. Therefore, let

$$\begin{aligned}
q_1 : \mathbb{F}_{2^n} &\rightarrow \mathbb{F}_2 : x \mapsto T_1(x^3 + x), \\
q_2 : \mathbb{F}_{2^n} &\rightarrow \mathbb{F}_2 : x \mapsto T_1(x^5 + x).
\end{aligned}$$

For $n \geq 3$ we have:

$$\begin{aligned}
F_2(n, 0, 0, 0) &= \frac{1}{2} \#\{x \in \mathbb{F}_{2^n} \mid T_1(x^3 + x) = T_1(x^5 + x) = 0\} \\
&= \frac{1}{2} \#\{x \in \mathbb{F}_{2^n} \mid q_1(x) = q_2(x) = 0\} \\
&= \frac{1}{4} (Z(q_1) + Z(q_2) + Z(q_1 + q_2) - 2^n),
\end{aligned}$$

by Lemma 5. Treating these zero-counts in turn, we have

$$\begin{aligned}
Z(q_1) &= \frac{1}{2} \#\{(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \mid y^2 + y = x^3 + x\} \\
&= \frac{1}{2} (\#E_1(\mathbb{F}_{2^n}) - 1) \\
&= \frac{1}{2} (2^n - (\sqrt{2}\omega_8^3)^n - (\sqrt{2}\omega_8^5)^n).
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
Z(q_2) &= \frac{1}{2} \#\{(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \mid y^2 + y = x^5 + x\} \\
&= \frac{1}{2} (\#H_2(\mathbb{F}_{2^n}) - 1) \\
&= \frac{1}{2} (2^n - (\sqrt{2}i)^n - (-\sqrt{2}i)^n - (\sqrt{2}\omega_8^3)^n - (\sqrt{2}\omega_8^5)^n),
\end{aligned}$$

since the Jacobian of $H_2/\mathbb{F}_2 : y^2 + y = x^5 + x$ is \mathbb{F}_2 -isogenous to the product of the two supersingular elliptic curves E_1 and E_2 [18, §4]. Finally, we have

$$\begin{aligned}
Z(q_1 + q_2) &= \frac{1}{2} \#\{(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \mid y^2 + y = x^5 + x^3\} \\
&= \frac{1}{2} (\#H_1(\mathbb{F}_{2^n}) - 1) \\
&= \frac{1}{2} (2^n - (\sqrt{2}\omega_{24}^5)^n - (\sqrt{2}\omega_{24}^{11})^n - (\sqrt{2}\omega_{24}^{13})^n - (\sqrt{2}\omega_{24}^{19})^n),
\end{aligned}$$

Combining these as per Lemma 5 gives $2^{n-3} + f_2(n, 0, 0, 0)$ as required. Furthermore, since $F_2(n, 0, 0, 0) + F_2(n, 0, 0, 1) = F_2(n, 0, 0)$, the formula for $f_2(n, 0, 0, 1)$ follows.

For the remaining two, let $F_2(n, 0, *, 0)$ be the count of those $a \in \mathbb{F}_{2^n}$ for which $T_1(a) = T_3(a) = 0$ and we do not mind what $T_2(a)$ is. Using the count for $Z(q_2)$ we have:

$$\begin{aligned}
F_2(n, 0, *, 0) &= \frac{1}{2} \#\{x \in \mathbb{F}_{2^n} \mid T_1(x^5 + x) = 0\} \\
&= \frac{1}{2} Z(q_2) \\
&= \frac{1}{4} (2^n - (\sqrt{2}i)^n - (-\sqrt{2}i)^n - (\sqrt{2}\omega_8^3)^n - (\sqrt{2}\omega_8^5)^n).
\end{aligned}$$

Since $F_2(n, 0, *, 0) = F_2(n, 0, 1, 0) + F_2(n, 0, 0, 0)$, we have

$$\begin{aligned}
&2^{n-2} - \frac{1}{4} ((\sqrt{2}i)^n + (-\sqrt{2}i)^n + (\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) \\
&= F_2(n, 0, 1, 0) + 2^{n-3} - \frac{1}{4} ((\sqrt{2}\omega_8^3)^n + (\sqrt{2}\omega_8^5)^n) - \frac{1}{8} ((\sqrt{2}i)^n + (-\sqrt{2}i)^n) \\
&\quad - \frac{1}{8} ((\sqrt{2}\omega_{24}^5)^n + (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n + (\sqrt{2}\omega_{24}^{19})^n).
\end{aligned}$$

Rearranging gives the formula for $f_2(n, 0, 1, 0)$. Finally, by $F_2(n, 0, 1, 0) + F_2(n, 0, 1, 1) = F_2(n, 0, 1)$, the formula for $f_2(n, 0, 1, 1)$ follows. \square

Our proof for all n is more direct than the amalgamation of the proofs given in [5] (n odd) and [19] (n even), and also explains in a simple way why the periodicity arises. Furthermore, for n odd one can derive the trace one formulae from the trace zero formulae using [5, Lemma 2.6], and thus all of the

results of that paper. One can also derive the trace one formulae for n odd directly, using the same curve based approach, noting that $T_1(1) = 1$ and expanding $T_2(x^2 + x + 1)$ and $T_3(x^2 + x + 1)$ and adjusting the definitions of q_1 and q_2 accordingly.

For completeness, we also include the Fourier-analysed formulae for the trace one cases, derived from [5] and [19].

Proposition 5. *For $n \geq 3$ we have*

$$\begin{aligned}
f_2(n, 1, 0, 0) &= \frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) - \frac{1+i}{8}(\sqrt{2}\omega_8^3)^n - \frac{1-i}{8}(\sqrt{2}\omega_8^5)^n \\
&\quad - \frac{i}{8}((\sqrt{2}\omega_{24}^5)^n - (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n - (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 1, 0, 1) &= -\frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) + \frac{1-i}{8}(\sqrt{2}\omega_8^3)^n + \frac{1+i}{8}(\sqrt{2}\omega_8^5)^n \\
&\quad + \frac{i}{8}((\sqrt{2}\omega_{24}^5)^n - (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n - (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 1, 1, 0) &= -\frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) + \frac{1+i}{8}(\sqrt{2}\omega_8^3)^n + \frac{1-i}{8}(\sqrt{2}\omega_8^5)^n \\
&\quad - \frac{i}{8}((\sqrt{2}\omega_{24}^5)^n - (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n - (\sqrt{2}\omega_{24}^{19})^n) \\
f_2(n, 1, 1, 1) &= \frac{1}{8}((\sqrt{2}i)^n + (-\sqrt{2}i)^n) - \frac{1-i}{8}(\sqrt{2}\omega_8^3)^n - \frac{1+i}{8}(\sqrt{2}\omega_8^5)^n \\
&\quad + \frac{i}{8}((\sqrt{2}\omega_{24}^5)^n - (\sqrt{2}\omega_{24}^{11})^n + (\sqrt{2}\omega_{24}^{13})^n - (\sqrt{2}\omega_{24}^{19})^n).
\end{aligned}$$

6 Computing $F_q(n, 0, 0, 0)$ using supersingular curves

In this and the following two sections we shall compute $F_q(n, 0, 0, 0)$ and Fourier-analyse the formulae. Recall that $q = 2^r$ with $r \geq 1$, which we use interchangeably as appropriate for the remainder of the paper. As in Sections 4 and 5, for $a \in \mathbb{F}_{q^n}$, if $T_1(a) = 0$ then $a = x^q + x$ for $x \in \mathbb{F}_{q^n}$. By Lemma 4(i) and (ii), we have $T_2(x^q + x) = T_1(x^{q+1} + x^2)$ and $T_3(x^q + x) = T_1(x^{2q+1} + x^{q+2})$ respectively. Therefore, let

$$\begin{aligned}
q_1 : \mathbb{F}_{q^n} &\rightarrow \mathbb{F}_q : x \mapsto T_1(x^{q+1} + x^2), \\
q_2 : \mathbb{F}_{q^n} &\rightarrow \mathbb{F}_q : x \mapsto T_1(x^{2q+1} + x^{q+2}).
\end{aligned}$$

We have the following generalisation of Lemma 5. As before, for a function $q_i : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q$ let $Z(q_i)$ denote the number of zeros of q_i , and for $w_i \in \mathbb{F}_q$ let $N(w_1, w_2)$ be the number of $x \in \mathbb{F}_{q^n}$ such that $q_i(x) = w_i$, for $i = 1, 2$.

Lemma 6. *Let $q_1, q_2 : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q$ be any functions. Then*

$$N(0, 0) = \frac{1}{q} (Z(q_1) + \sum_{\alpha \in \mathbb{F}_q} Z(\alpha q_1 + q_2) - q^n)$$

Proof. We have the following identities:

$$q^n = \sum_{\alpha, \beta \in \mathbb{F}_q} N(\alpha, \beta), \quad Z(q_1) = \sum_{\beta \in \mathbb{F}_q} N(0, \beta), \quad \text{and} \quad \sum_{\alpha \in \mathbb{F}_q} Z(\alpha q_1 + q_2) = \sum_{\alpha, \beta \in \mathbb{F}_q} N(\beta, \alpha\beta).$$

We have

$$\sum_{\alpha, \beta \in \mathbb{F}_q} N(\alpha, \beta) = \sum_{\beta \in \mathbb{F}_q} N(0, \beta) + \sum_{\beta \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q^\times} N(\alpha, \beta) \quad (8)$$

$$\begin{aligned} &= \sum_{\beta \in \mathbb{F}_q} N(0, \beta) + \sum_{\beta \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q^\times} N(\alpha, \alpha\beta) \\ &= \sum_{\beta \in \mathbb{F}_q} N(0, \beta) + \sum_{\alpha, \beta \in \mathbb{F}_q} N(\beta, \alpha\beta) - qN(0, 0), \end{aligned} \quad (9)$$

where the equality between the second terms on the r.h.s. of lines (8) and (9) follow from the fact that for each $\alpha \in \mathbb{F}_q^\times$ we have $\{N(\alpha, \beta)\}_{\beta \in \mathbb{F}_q} = \{N(\alpha, \alpha\beta)\}_{\beta \in \mathbb{F}_q}$. Rearranging for $N(0, 0)$ gives the stated result. \square

Informed by Lemma 6 we employ the following curves, for $\alpha \in \mathbb{F}_q$:

$$\begin{aligned} C : y^q + y &= x^{q+1} + x^2, \\ C_\alpha : y^q + y &= x^{2q+1} + x^{q+2} + \alpha(x^{q+1} + x^2). \end{aligned}$$

Observe that for $\alpha \in \mathbb{F}_q^\times$ we have the \mathbb{F}_q -isomorphism $\sigma : C_\alpha \rightarrow C_1 : (x, y) \mapsto (\alpha x, \alpha^3 y)$ and thus their number of rational points over any extension of \mathbb{F}_q are equal. Hence the formula in Lemma 6 simplifies to:

$$N(0, 0) = \frac{1}{q} (Z(q_1) + Z(q_2) + (q-1)Z(q_1 + q_2) - q^n). \quad (10)$$

In the following we therefore focus on these three curves:

$$\begin{aligned} C_1/\mathbb{F}_q : y^q + y &= x^{q+1} + x^2, \\ C_2/\mathbb{F}_q : y^q + y &= x^{2q+1} + x^{q+2}, \\ C_3/\mathbb{F}_q : y^q + y &= x^{2q+1} + x^{q+2} + x^{q+1} + x^2. \end{aligned}$$

By (10) one can express $F_q(n, 0, 0, 0)$ in terms of the number of points of C_1, C_2 and C_3 over \mathbb{F}_{q^n} . We shall compute these numbers, which incidentally proves the supersingularity of the three curves (there are many other ways to do so too). In the rest of this section we detail some preliminary notions and results.

6.1 Simplifying the point counting on C_1, C_2 and C_3

In this subsection we show that counting the number of points on the curves C_1, C_2 and C_3 over \mathbb{F}_{q^n} can be simplified by obtaining a partial decomposition of the Jacobian of Artin-Schreier curves. Let $\overline{\mathbb{F}_2}(x)$ denote the rational function field over $\overline{\mathbb{F}_2}$ which is the algebraic closure of the binary field \mathbb{F}_2 . An Artin-Schreier curve is

$$C : y^q + y = f(x)$$

where $q = 2^r$ and $f(x) \in \overline{\mathbb{F}_2}(x)$ is not a constant and has no pole of an even order. When $f(x)$ is a polynomial this is equivalent to assuming that $f(x)$ is of an odd degree. If P_1, P_2, \dots, P_m are the poles of $f(x)$ of order e_1, e_2, \dots, e_m respectively, then the genus of C is

$$g_C = \frac{q-1}{2} \left(\sum_{i=1}^m (e_i + 1) - 2 \right). \quad (11)$$

If $f(x)$ is a polynomial of an odd degree d , then the genus of C simply is

$$g_C = \frac{(q-1)(d-1)}{2}. \quad (12)$$

We obtain a partial decomposition of the Jacobian of the Artin-Schreier curve C by studying the set of automorphisms of the l.h.s. $y^q + y$ only. Computing the relevant quotient curves and applying a theorem due to Kani and Rosen [9] allows one to then infer a relation between the number of points on the original curve and on the quotient curves.

For any positive integer m , let $\text{Tr}_m : \mathbb{F}_{2^m} \rightarrow \mathbb{F}_2$ be the absolute trace function $a \mapsto a + a^2 + a^{2^2} + \dots + a^{2^{m-1}}$. We need the following lemma to establish the main result of this subsection.

Lemma 7. *Let α be a nonzero element of \mathbb{F}_{2^r} , and let $H_\alpha = \{x \in \mathbb{F}_{2^r} \mid \text{Tr}_r(\alpha x) = 0\}$. Then*

- (i) H_α is an additive subgroup of \mathbb{F}_{2^r} when \mathbb{F}_{2^r} is viewed as an additive group,
- (ii) half the elements of \mathbb{F}_{2^r} are in H_α , i.e., $\#H_\alpha = 2^{r-1}$, and
- (iii) $\prod_{\beta \in \mathbb{F}_{2^r} \setminus H_\alpha} \beta = \alpha^{2^{r-1}-1}$.

Proof. It is easy to check the first part of the claim. To show (ii), notice that the traces of half the elements of \mathbb{F}_{2^r} are zero, and since α is an element of \mathbb{F}_{2^r} , the map $\psi : H_\alpha \rightarrow \mathbb{F}_{2^r}$ taking x to αx induces a bijection between elements of H_α and elements z of \mathbb{F}_{2^r} for which $\text{Tr}_r(z) = 0$. To prove (iii), we first notice that

$$\text{Tr}_r(\alpha x) = \alpha x + (\alpha x)^2 + \dots + (\alpha x)^{2^{r-1}},$$

and hence $\text{Tr}_r(\alpha x)$ is a polynomial of degree 2^{r-1} which has 2^{r-1} roots in H_α . Thus we have

$$\text{Tr}_r(\alpha x) = \alpha^{2^{r-1}} \prod_{\gamma \in H_\alpha} (x + \gamma).$$

This implies that

$$\prod_{\substack{\gamma \in H_\alpha \\ \gamma \neq 0}} \gamma = \alpha^{-2^{r-1}+1}.$$

Now the claim follows from the fact that the product of all non-zero elements of any binary finite field is 1. \square

Let $\alpha \in \mathbb{F}_{2^r}^\times$. It is easy to check that α induces the involution ϕ_α of C where ϕ_α is given by

$$\phi_\alpha : (x, y) \rightarrow (x, y + \alpha),$$

and hence the group of automorphisms of C , $\text{Aut}(C)$, has a subgroup isomorphic to the additive group of \mathbb{F}_{2^r} . Thus it follows from Lemma 7 that H_α can be viewed as a subgroup of $\text{Aut}(C)$. In the following for every $\alpha \in \mathbb{F}_{2^r}$, we compute a plane model for C/H_α .

Lemma 8. *With the notations as above we have*

$$C/H_\alpha : y_\alpha^2 + \alpha^{2^{r-1}-1} y_\alpha = f(x).$$

Proof. In order to compute a plane model for C/H_α , we need to find two elements in the function field $\overline{\mathbb{F}_2}(x, y)$ which are invariant under the action of H_α , and then find an algebraic relation between the two elements. Since H_α is an additive subgroup of \mathbb{F}_{2^r} , it follows that

$$y_\alpha = \prod_{\gamma \in H_\alpha} (y + \gamma)$$

is invariant under the action of H_α . So we choose y_α to be one of the elements that we are looking for and we take the other element to be simply x .

Now we claim that

$$y^q + y = y_\alpha^2 + \alpha^{2^{r-1}-1} y_\alpha.$$

In order to prove this claim, we need to prove that any element $\delta \in \mathbb{F}_{2^r}$ is a root of the right hand side of the above equation. If $\delta \in H_\alpha$, then the claim is trivial. Now suppose that $\delta \in \mathbb{F}_{2^r} \setminus H_\alpha$. First notice that

$$y_\alpha(\delta) + \alpha^{2^{r-1}-1} = \prod_{\gamma \in H_\alpha} (\delta + \gamma) + \alpha^{2^{r-1}-1}.$$

Furthermore since H_α is an additive subgroup of \mathbb{F}_{2^r} of index two, we have

$$\prod_{\gamma \in H_\alpha} (\delta + \gamma) + \alpha^{2^{r-1}-1} = \left(\prod_{\beta \in \mathbb{F}_{2^r} \setminus H_\alpha} \beta \right) + \alpha^{2^{r-1}-1},$$

and hence the claim follows from Lemma 7 part (iii).

We shall use the following theorem due to Kani and Rosen, which is from Theorem C in [9].

Theorem 3. *Let C be a smooth, projective, absolutely irreducible curve defined over an arbitrary field K , and let $H_1, \dots, H_t < \text{Aut}(C)$ be (finite) subgroups with $H_i \cdot H_j = H_j \cdot H_i$ for all i, j . Furthermore let g_{ij} denote the genus of the quotient curve $C/(H_i \cdot H_j)$. Then if $g_{ij} = 0$ for $2 \leq i < j \leq t$ and if*

$$g_C = g_{C/H_2} + \dots + g_{C/H_t},$$

then we have (by taking $H_1 = \{1\}$ above) an isogeny of Jacobians:

$$J_C \sim J_{C/H_2} \times \dots \times J_{C/H_t}.$$

Applying the above theorem we get the following result.

Theorem 4. *Let C denote the Artin-Schrier curve $y^{2^r} + y = f(x)$, and for every $\alpha \in \mathbb{F}_{2^r}^\times$ let C_α denote the curve $y_\alpha^2 + y_\alpha = \alpha f(x)$. Furthermore let H_α be as in Lemma 7. Then we have*

$$J_C \sim \prod_{\alpha \in \mathbb{F}_{2^r}^\times} J_{C/H_\alpha} \sim \prod_{\alpha \in \mathbb{F}_{2^r}^\times} J_{C_\alpha}.$$

Proof. For every pair of distinct elements α and β in $\mathbb{F}_{2^r}^\times$, we trivially have $H_\alpha \cdot H_\beta = H_\beta \cdot H_\alpha$ and furthermore $(H_\alpha \cdot H_\beta) = \mathbb{F}_{2^r}$ where we are considering \mathbb{F}_{2^r} as an additive group. Notice that for every nonzero α , H_α is a subgroup of index two of \mathbb{F}_{2^r} . It follows that the genus $g_{\alpha\beta}$ of the curve $C/(H_\alpha \cdot H_\beta)$ is zero. The fact that

$$g_C = \sum_{\alpha \in \mathbb{F}_{2^r}^\times} g_{C/H_\alpha}$$

follows from (11). Thus we have

$$J_C \sim \prod_{\alpha \in \mathbb{F}_{2^r}^\times} J_{C/H_\alpha}.$$

The rest of the claim follows from the fact that by the change of variable $y_\alpha \mapsto \alpha^{2^{r-1}-1} y_\alpha$, the curve equations for C/H_α becomes $C_\alpha : y_\alpha^2 + y_\alpha = \alpha f(x)$.

The following is an immediate corollary of the above theorem as the above theorem implies that the L-polynomial of the curve C is equal to product of the L-polynomials of the curves C_α .

Corollary 1. *Let C and C_α be as defined above. Then*

$$\#C(\mathbb{F}_{q^n}) - \sum_{\alpha \in \mathbb{F}_q^\times} \#C_\alpha(\mathbb{F}_{q^n}) = (q^n + 1)(q - 1) - 1.$$

7 Computing the number of rational points on C_1 , C_2 and C_3

With a view to applying Lemma 6 to compute $F_q(n, 0, 0, 0)$, define

$$\begin{aligned} q_1(x) &:= x^{2^r+1} + x^2, \\ q_2(x) &:= x^{2^{r+1}+1} + x^{2^r+2}, \\ q_3(x) &:= q_1(x) + q_2(x), \end{aligned}$$

where $r \geq 1$ is an integer. Let $n \geq 1$ be an integer. In this section, our aim is to find the number of rational points on the curves

$$C_i : y^{2^r} + y = q_i(x)$$

on $\mathbb{F}_{2^{rn}}$ for all $i = 1, 2, 3$. In order to find this we will first try to find the number of rational points on the curves

$$C_{i,\alpha} : y^2 + y = \alpha q_i(x)$$

on $\mathbb{F}_{2^{rn}}$ for all $\alpha \in \mathbb{F}_{2^r}^\times$ and for all $i = 1, 2, 3$. We will then apply Corollary 1 to find the number of points on C_i .

Proposition 6. *Let p be an odd prime divisor of n . Then $\#C_{i,\alpha}(\mathbb{F}_{2^{rn}}) - \#C_{i,\alpha}(\mathbb{F}_{2^{r(n/p)}}) \equiv 0 \pmod{p}$ for all $\alpha \in \mathbb{F}_{2^r}^\times$ and for all $i = 1, 2, 3$.*

Proof. We have $\text{Tr}_{rn}(\alpha q_i(x)) = p \cdot \text{Tr}_{r(n/p)}(\alpha q_i(x)) = \text{Tr}_{r(n/p)}(\alpha q_i(x))$ for all $x \in \mathbb{F}_{2^{r(n/p)}}$, for all $\alpha \in \mathbb{F}_{2^r}^\times$ and for all $i = 1, 2, 3$. Let $\beta \in \mathbb{F}_{2^{rn}} - \mathbb{F}_{2^{r(n/p)}}$. Then $\text{Tr}_{rn}(\alpha q_i(x^{2^r})) = \text{Tr}_{rn}(\alpha q_i(x))^{2^r} = \text{Tr}_{rn}(\alpha q_i(x))$ for all $\alpha \in \mathbb{F}_{2^r}^\times$ and for all $i = 1, 2, 3$. Since order of $\text{Gal}(\mathbb{F}_{2^{rn}}/\mathbb{F}_{2^{r(n/p)}})$ conjugates of β , which is $\{\beta, \beta^{2^r}, \dots, \beta^{2^{r(p-1)}}\}$, is p , we have $\#C_{i,\alpha}(\mathbb{F}_{2^{rn}}) - \#C_{i,\alpha}(\mathbb{F}_{2^{r(n/p)}}) \equiv 0 \pmod{p}$. \square

The number $\#C_{i,\alpha}(\mathbb{F}_{2^{rn}})$ can be written as

$$\#C_{i,\alpha}(\mathbb{F}_{2^{rn}}) = (2^{rn} + 1) + \Lambda_{i,\alpha}(n)2^{(rn+w_{i,\alpha}(n))/2} \quad (13)$$

where $w_{i,\alpha}(n)$ is the dimension of radical of $\text{Tr}_{rn}(\alpha q_i(x))$ by Proposition 2. We call $\Lambda_{i,\alpha}(n) \in \{-1, 0, 1\}$ as sign of $\#C_{i,\alpha}(\mathbb{F}_{2^{rn}})$.

Corollary 2. *Let $n \geq 1$ be an integer such that $n = 2^u m$ where $u \geq 1$ be an integer and m is odd. Assume $\#C_{i,\alpha}(\mathbb{F}_{2^{rn}})$ and dimension of radical of $\text{Tr}_{rn'}(\alpha q_i(x))$ are known where n' equals to $2^u m'$ and m' is odd. Then $\#C_{i,\alpha}(\mathbb{F}_{2^{rn'}})$ is known.*

Proof. We can assume $m = 1$ and $m' = p > 2$. Since dimension of radical of $\text{Tr}_{r2^u p}(\alpha q_i(x))$ is known, It is enough to find sign of $C_{i,\alpha}(\mathbb{F}_{r2^u p})$ by Equation 13. Since $\#C_{i,\alpha}(\mathbb{F}_{r2^u})$ is known and $\#C_{i,\alpha}(\mathbb{F}_{r2^u p}) \equiv \#C_{i,\alpha}(\mathbb{F}_{r2^u}) \pmod{p}$, the sign of $C_{i,\alpha}(\mathbb{F}_{r2^u p})$ is also known. \square

Note that Proposition 1 and Corollary 2 imply that in order to find the number of rational points of $C_{i,\alpha}$ on $\mathbb{F}_{2^{rn}}$ for all $n \geq 1$, it is enough to find the number of rational points of $C_{i,\alpha}$ on $\mathbb{F}_{2^{r2^u}}$ where $0 \leq u \leq v$ and v is the minimal integer for which $C(\mathbb{F}_{2^{r2^v}})$ is minimal, and also the dimension of the radical of $\text{Tr}_{rn}(\alpha q_i(x))$ for all $n \geq 1$.

7.1 Number of $\mathbb{F}_{2^{rn}}$ -rational points on the curve $C_1 : y^{2^r} + y = q_1(x)$

In this section we will find the number of rational points on the curves

$$C_1 : y^{2^r} + y = x^{2^r+1} + x^2$$

on $\mathbb{F}_{2^{rn}}$ for all $r, n \in \mathbb{Z}^+$. In order to find this we will first try to find the number of rational points on the curves

$$C_{1,\alpha} : y^2 + y = \alpha(x^{2^r+1} + x^2)$$

on $\mathbb{F}_{2^{rn}}$ for all $\alpha \in \mathbb{F}_{2^r}^\times$, and we will then apply Corollary 1. We will see that the number of points is the same for each α .

Lemma 9. *Let $B_1(x, y) = \text{Tr}_{rn}(q_1(x + y) + q_1(x) + q_1(y))$ and let the radical of $B_1(x, y)$ be $W_1(n) = \{x \in \mathbb{F}_{2^{rn}} \mid B_1(x, y) = 0 \text{ for all } y \in \mathbb{F}_{2^{rn}}\}$. Denote the dimension of radical of $B_1(x, y)$ over $\mathbb{F}_{2^{rn}}$ as $w_r(n)$, then we have*

$$w_r(n) = \begin{cases} r & \text{if } n \text{ is odd} \\ 2r & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $B_1(x, y) = \text{Tr}_{rn}(q_1(x + y) + q_1(x) + q_1(y))$. Then we have

$$B_1(x, y) = \text{Tr}_{rn}(y^{2^r}(x^{2^{2r}} + x))$$

and the radical of $B_1(x, y)$ is

$$W_1(n) = \{x \in \mathbb{F}_{2^{rn}} \mid B_1(x, y) = 0 \text{ for all } y \in \mathbb{F}_{2^{rn}}\} = \mathbb{F}_{2^{rn}} \cap \mathbb{F}_{2^{2r}} = \mathbb{F}_{2^{r(2,n)}}.$$

□

Lemma 10. *Let the notation be above. $\text{Tr}_{2r}(q_1(W_1(2))) = \mathbb{F}_2$ and $\text{Tr}_{2r}(q_1(W_1(n))) = 0$ for $4 \mid n$.*

Proof. Suppose $n = 2$. Since $x^{2^r+1} \in \mathbb{F}_{2^r}$ for all $x \in \mathbb{F}_{2^{2r}}$ we have $\text{Tr}_{2r}(x^{2^r+1}) = 0$ for all $x \in \mathbb{F}_{2^{2r}}$. Therefore

$$\text{Tr}_{2r}(x^{2^r+1} + x^2) = \text{Tr}_{2r}(x).$$

Now suppose $4 \mid n$. Since $W_1(n) = \mathbb{F}_{2^{2r}}$, for all $x \in W_1(n)$ we have

$$\text{Tr}_{rn}(x^{2^r+1} + x^2) = 0$$

for all $x \in W_1(n)$.

□

Theorem 5. *If r is odd, then we have*

$$\#C_{1,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 7 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+1)/2} & \text{if } n \equiv 3, 5 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 2, 6 \pmod{8}, \\ (2^{rn} + 1) + 2^{r(n+2)/2} & \text{if } n \equiv 4 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2} & \text{if } n \equiv 0 \pmod{8} \end{cases}$$

if r is even, then we have

$$\#C_{1,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 3 \pmod{4}, \\ (2^{rn} + 1) & \text{if } n \equiv 2 \pmod{4}, \\ (2^{rn} + 1) - 2^{r(n+2)/2} & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Proof. In order to prove this theorem, we will find the number of rational points of $C_{1,1}$ on \mathbb{F}_{2^r} , $\mathbb{F}_{2^{2r}}$, $\mathbb{F}_{2^{4r}}$ and $\mathbb{F}_{2^{8r}}$. Then we will apply Proposition 1 and Proposition 6 for finding the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{rn}}$ for any $n \geq 1$.

Since $x^{2^r+1} + x^2 = 0$ for all $x \in \mathbb{F}_{2^r}$, the number of rational points of $C_{1,1}$ on \mathbb{F}_{2^r} is $(2^r + 1) + 2^r$. Therefore, the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{rn}}$ is $(2^{rn} + 1) + 2^{r(n+1)/2}$ if $n \equiv 1, 7 \pmod{8}$ and the number of rational points of $C_{1,1}$ on \mathbb{F}_{2^r} is $(2^{rn} + 1) + 2^{r(n+1)/2}$ if $n \equiv 3, 5 \pmod{8}$ by Proposition 6 and Lemma 9.

Since $\text{Tr}_{2^r}(q_1(W_1(2))) = \mathbb{F}_2$ by Lemma 10, we have the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{2r}}$ is $2^{2r} + 1$. Therefore, the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{rn}}$ is $2^{rn} + 1$ if $n \equiv 2, 6 \pmod{8}$ by Proposition 6 and Lemma 9.

Since $\text{Tr}_{4r}(q_1(W_1(4))) = 0$ by Lemma 10, we have the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{2r}}$ is $(2^{2r} + 1) \pm 2^{r(n+2)/2}$. Since genus of $C_{1,1}$ is 2^{r-1} we have $C_1(\mathbb{F}_{2^{4r}})$ is maximal or minimal by Proposition 1. Since we cannot write 1 as a \mathbb{Q} linear combination of $\{w \in \mathbb{C} \mid w^4 = -1\}$ by Lemma 3, and since we cannot write $\sqrt{2}$ as a \mathbb{Q} linear combination of $\{w \in \mathbb{C} \mid w^4 = 1\}$, we have $C_{1,1}(\mathbb{F}_{2^{4r}})$ is maximal and $C_{1,1}(\mathbb{F}_{2^{8r}})$ is minimal by Proposition 1 if r is odd, and $C_{1,1}(\mathbb{F}_{2^{4r}})$ is minimal if r is even. Therefore if r is odd, the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{rn}}$ is $(2^{rn} + 1) + 2^{r(n+2)/2}$ if $n \equiv 4 \pmod{8}$ and $(2^{rn} + 1) - 2^{r(n+2)/2}$ if $n \equiv 0 \pmod{8}$; if r is even, the number of rational points of $C_{1,1}$ on $\mathbb{F}_{2^{rn}}$ is $(2^{rn} + 1) - 2^{r(n+2)/2}$ if $n \equiv 0, 4 \pmod{8}$ by Proposition 6 and Lemma 9.

Note that since the map $(x, y) \rightarrow (\alpha^{-1/2}x, y)$ gives an isomorphism between $C_{1,\alpha} \rightarrow C_{1,1}$ where $\alpha \in \mathbb{F}_{2^r}^\times$, their number of rational points equal in any extension of \mathbb{F}_{2^r} . \square

Using Corollary 1 and Theorem 5, Table 3 in the appendix presents the number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_1 : y^{2^r} + y = q_1(x)$.

7.2 Number of $\mathbb{F}_{2^{rn}}$ -rational points on the curve $C_2 : y^{2^r} + y = q_2(x)$

In this section we will find the number of $\mathbb{F}_{2^{rn}}$ -rational points on the curve

$$C_2 : y^{2^r} + y = x^{2^{r+1}+1} + x^{2^r+2}$$

for all $r, n \in \mathbb{Z}^+$. In order to find this we will first try to find the number of rational points on the curves

$$C_{2,\alpha} : y^2 + y = \alpha(x^{2^{r+1}+1} + x^{2^r+2})$$

on $\mathbb{F}_{2^{rn}}$ for all $\alpha \in \mathbb{F}_{2^r}^\times$, and we will then apply Corollary 1. We will see that the number of points depends on whether α is a cube, for odd r .

Lemma 11. *Let $B_{2,\alpha}(x, y) = \text{Tr}_{rn}(\alpha(q_2(x+y) + q_2(x) + q_2(y)))$ where $\alpha \in \mathbb{F}_{2^r}$ and let the radical of $B_{2,\alpha}(x, y)$ be $W_1(n) = \{x \in \mathbb{F}_{2^{rn}} \mid B_{2,\alpha}(x, y) = 0 \text{ for all } y \in \mathbb{F}_{2^{rn}}\}$. Denote the dimension of radical of $B_1(x, y)$ over $\mathbb{F}_{2^{rn}}$ as $w_r(n)$. If $\alpha \in \{x^3 \mid x \in \mathbb{F}_{2^r}\}$, then we have*

$$w_{r,\alpha}(n) = \begin{cases} r & \text{if } n \text{ is odd,} \\ 2r & \text{if } n \equiv 2 \pmod{4}, \\ 2r + 2 & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

If $\alpha \notin \{x^3 \mid x \in \mathbb{F}_{2^r}\}$, then we have

$$w_{r,\alpha}(n) = \begin{cases} r & \text{if } (n, 6) = 1, \\ 2r & \text{if } (n, 6) = 2, \\ r + 2 & \text{if } (n, 6) = 3, \\ 2r + 2 & \text{if } (n, 6) = 6. \end{cases}$$

Proof. Let $B_{2,\alpha}(x, y) = \text{Tr}_{rn}(\alpha(q_2(x+y) + q_2(x) + q_2(y)))$ where $\alpha \in \mathbb{F}_{2^r}$. Then we have

$$B_{2,\alpha}(x, y) = \text{Tr}_{rn}(\alpha y^{2^{r+1}}[(x + \alpha x^4) + (x + \alpha x^4)^{2^{2r}}])$$

and the radical of $B_{2,\alpha}(x, y)$ is

$$W_{2,\alpha}(n) = \{x \in \mathbb{F}_{2^{rn}} \mid (x + \alpha x^4) + (x + \alpha x^4)^{2^{2r}} = 0\}.$$

This can be rewritten as (multiplying by α^3 and transforming $\alpha x \rightarrow x$)

$$W_{2,\alpha}(n) = \{x \in \mathbb{F}_{2^{rn}} \mid (x^4 + \alpha^2 x) + (x^4 + \alpha^2 x)^{2^{2r}} = 0 \in \mathbb{F}_{2^{2r}}\}.$$

Let $x^4 + \alpha^2 x = x(x+a)(x+b)(x+c)$ where $a, b, c \in \overline{\mathbb{F}_{2^r}}$. Then

$$(x^4 + \alpha^2 x) + (x^4 + \alpha^2 x)^{2^{2r}} = (x^{2^{2r}} + x)(x^{2^{2r}} + x + a)(x^{2^{2r}} + x + b)(x^{2^{2r}} + x + c).$$

Therefore $w_{r,\alpha}(n) = \frac{1}{2}\#\{u : u \in \{0, a, b, c\} \text{ and } u \in \mathbb{F}_{2^{rn}}\} + r(n, 2)$. Moreover if α^2 (or α) is not a cube in \mathbb{F}_{2^r} then $x^3 + \alpha^2$ is irreducible over \mathbb{F}_{2^r} and hence all roots of $x^3 + \alpha^2$ are in $\mathbb{F}_{2^{3r}}$, if α^2 (or α) is a cube in \mathbb{F}_{2^r} then $x^{4r} + x = 0$ for all $x \in W_{2,\alpha}(n)$ and hence all roots of $x^3 + \alpha^2$ are in $\mathbb{F}_{2^{4r}}$. \square

Lemma 12. *Let notation be as above. Then we have $\text{Tr}_{2r}(\alpha q_2(W_{2,\alpha}(n))) = 0$ and therefore $\#C_{2,\alpha}(\mathbb{F}_{2^{2r}}) - (2^{2r} + 1) \neq 0$.*

Proof. Since $\alpha(x^{2^{r+1}+1} + x^{2^{r+2}}) \in \mathbb{F}_{2^r}$ for all $x \in \mathbb{F}_{2^{2r}}$ and for all $\alpha \in \mathbb{F}_{2^r}$, we have

$$\text{Tr}_{2r}(\alpha(x^{2^{r+1}+1} + x^{2^{r+2}})) = 0$$

for all $x \in \mathbb{F}_{2^{2r}}$ and for all $\alpha \in \mathbb{F}_{2^r}$. Therefore $\#C_{2,\alpha}(\mathbb{F}_{2^{2r}}) - (2^{2r} + 1) \neq 0$. \square

Theorem 6. *If r is odd, then we have*

$$\#C_{2,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 7 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+1)/2} & \text{if } n \equiv 3, 5 \pmod{8}, \\ (2^{rn} + 1) + 2^{r(n+2)/2} & \text{if } n \equiv 2, 6 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 4 \pmod{8} \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

If r is even and $\alpha \in \{x^3 \mid x \in \mathbb{F}_{2^r}\}$, then we have

$$\#C_{2,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 3 \pmod{4}, \\ (2^{rn} + 1) + 2^{r(n+2)/2} & \text{if } n \equiv 2 \pmod{4}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

If r is even and $\alpha \notin \{x^3 \mid x \in \mathbb{F}_{2^r}\}$, then we have

$$\#C_{2,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } (n, 6) = 1, \\ (2^{rn} + 1) + 2^{r(n+2)/2} & \text{if } (n, 6) = 2, \\ (2^{rn} + 1) - 2^{r(n+1)/2+1} & \text{if } (n, 3) = 3, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } (n, 6) = 6. \end{cases}$$

Proof. We will prove this theorem in same manner of Theorem 5. Note that the genus of $C_{2,\alpha}$ is 2^r for all $\alpha \in \mathbb{F}_{2^r}^\times$.

Assume r is odd. Suppose $C_{2,\alpha}(\mathbb{F}_{2^{4r}})$ is maximal. Then it requires $\#C_{2,\alpha}(\mathbb{F}_{2^{2r}}) - (2^{2r} + 1) = 0$, but it is not possible by Lemma 12. Suppose $C_{2,\alpha}(\mathbb{F}_{2^{4r}})$ is minimal curve. Then we cannot write $\sqrt{2}$ as a \mathbb{Q} linear combination of $\{\eta \in \mathbb{C} \mid \eta^4 = 1\}$. Therefore $\#C_{2,\alpha}(\mathbb{F}_{2^{4r}}) - (2^{4r} + 1) = 0$. Furthermore, since $W_{1,\alpha}(n) \subset \mathbb{F}_{2^{4r}}$, we have $C_{2,\alpha}(\mathbb{F}_{2^{8r}})$ is maximal or minimal. Since $\sqrt{2}$ cannot be written as a \mathbb{Q} linear combination of $\{\eta \in \mathbb{C} \mid \eta^8 = -1\}$ by Lemma 3, the curve $C_{2,\alpha}(\mathbb{F}_{2^{8r}})$ must be minimal.

Assume r is even. Then $\alpha q_2(wx) = \alpha q_2(w^2x) = \alpha q^2(x)$ for all $x \in \mathbb{F}_{2^{rn}}$ and for all $\alpha \in \mathbb{F}_{2^r}^\times$ where $w^2 + w + 1 = 0$ and therefore $\#C_{2,\alpha}(\mathbb{F}_{2^{rn}}) \equiv 0 \pmod{3}$. Since we know the dimensions $w_{r,\alpha}(n)$ for all $\alpha \in \mathbb{F}_{2^r}^\times$ and $n \geq 1$, one can easily find the signs of $C_{2,\alpha}(\mathbb{F}_{2^{rn}})$ for all $\alpha \in \mathbb{F}_{2^r}^\times$ and $n \geq 1$. \square

Using Corollary 1 and Theorem 6, Table 4 in the appendix presents the number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_2 : y^{2^r} + y = q_2(x)$.

We remark that Moisio and Ranto [15] have counted $F_q(n, 0, *, t_3)$ for $t_3 \in \mathbb{F}_q = \mathbb{F}_{2^r}$. Their methods are different, and do not yield the same type of general formula as here.

7.3 Number of $\mathbb{F}_{2^{rn}}$ -rational points on the curve $C_3 : y^{2^r} + y = q_3(x)$

In this section we will find the number of rational points on the curve

$$C_3 : y^{2^r} + y = x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2$$

on $\mathbb{F}_{2^{rn}}$ for all $r, n \in \mathbb{Z}^+$. In order to find this we will first try to find the number of rational points on the curves

$$C_{3,\alpha} : y^2 + y = \alpha(x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2)$$

on $\mathbb{F}_{2^{rn}}$ for all $\alpha \in \mathbb{F}_{2^r}^\times$.

Lemma 13. Let $B_{3,\alpha}(x, y) = \text{Tr}_{rn}(\alpha(q_3(x+y) + q_3(x) + q_3(y)))$ where $\alpha \in \mathbb{F}_{2^r}$ and let the radical of $B_{3,\alpha}(x, y)$ be $W_{3,\alpha}(n) = \{x \in \mathbb{F}_{2^{rn}} \mid B_{3,\alpha}(x, y) = 0 \text{ for all } y \in \mathbb{F}_{2^{rn}}\}$. Denote the dimension of radical of $B_{3,\alpha}(x, y)$ over $\mathbb{F}_{2^{rn}}$ as $w_r(n)$. If $x^3 + x + \alpha^{-1}$ has one root or three roots in \mathbb{F}_{2^r} , then we have

$$w_{r,\alpha}(n) = \begin{cases} r & \text{if } n \text{ is odd,} \\ 2r & \text{if } n \equiv 2 \pmod{4}, \\ 2r + 2 & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

If $x^3 + x + \alpha^{-1}$ has no root in \mathbb{F}_{2^r}

$$w_{r,\alpha}(n) = \begin{cases} r & \text{if } (n, 6) = 1, \\ 2r & \text{if } (n, 6) = 2, \\ r + 2 & \text{if } (n, 6) = 3, \\ 2r + 2 & \text{if } (n, 6) = 6. \end{cases}$$

Proof. Let $B_{3,\alpha}(x, y) = \text{Tr}_{rn}(\alpha(q_3(x+y) + q_3(x) + q_3(y)))$ where $\alpha \in \mathbb{F}_{2^r}$. Then we have

$$B_{3,\alpha}(x, y) = \text{Tr}_{rn}(\alpha^2 y^{2^{r+1}} [(\alpha^{-1}x + x^2 + x^4) + (\alpha^{-1}x + x^2 + x^4)^{2^{2r}}])$$

and the radical of $B_{3,\alpha}(x, y)$ is

$$W_{3,\alpha}(n) = \{x \in \mathbb{F}_{2^{rn}} \mid (\alpha^{-1}x + x^2 + x^4) + (\alpha^{-1}x + x^2 + x^4)^{2^{2r}} = 0\}.$$

Let $x^4 + x^2 + \alpha^{-1}x = x(x+a)(x+b)(x+c)$ where $a, b, c \in \overline{\mathbb{F}_{2^r}}$. Then

$$(x^4 + x^2 + \alpha^{-1}x) + (x^4 + x^2 + \alpha^{-1}x)^{2^r} = (x^{2^r} + x)(x^{2^r} + x + a)(x^{2^r} + x + b)(x^{2^r} + x + c).$$

Therefore $w_{r,\alpha}(n) = \frac{1}{2}\#\{u : u \in \{0, a, b, c\} \text{ and } u \in \mathbb{F}_{2^{rn}}\} + r(n, 2)$. Moreover if $x^3 + x + \alpha^{-1}$ is irreducible over \mathbb{F}_{2^r} , then all roots of $x^3 + x + \alpha^{-1}$ are in $\mathbb{F}_{2^{3r}}$, if $x^3 + x + \alpha^{-1}$ has one root in \mathbb{F}_{2^r} and two roots in $\mathbb{F}_{2^{2r}}$; or three roots in \mathbb{F}_{2^r} then $x^{4r} + x = 0$ for all $x \in W_{2,\alpha}(n)$ and hence all roots of $x^3 + x + \alpha^{-1}$ are in $\mathbb{F}_{2^{4r}}$. \square

Lemma 14. *Let $x^{2^{2r}} + x + b = 0$ where $b \in \mathbb{F}_{2^{2r}}$. Then*

$$\text{Tr}_{4r}(\alpha(x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2)) = \begin{cases} 0 & \text{if } b \in \mathbb{F}_{2^{2r}} \\ 1 & \text{if } b \in \mathbb{F}_{2^{2r}} \setminus \mathbb{F}_{2^r} \end{cases}$$

where $\alpha \in \mathbb{F}_{2^r}^\times$. Therefore, if $x^3 + x + \alpha^{-1}$ has zero root or three roots in \mathbb{F}_{2^r} , then $\text{Tr}_{4r}(\alpha q_3(W_{3,\alpha})) = 0$ and if $x^3 + x + \alpha^{-1}$ has one root in \mathbb{F}_{2^r} , then $\text{Tr}_{4r}(\alpha q_3(W_{3,\alpha})) = \mathbb{F}_2$.

Proof. We expand the trace

$$\begin{aligned} & \text{Tr}_{4r}(\alpha(x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2)) \\ &= \text{Tr}_{2r}(\alpha(x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2) + (\alpha(x^{2^{r+1}+1} + x^{2^r+2} + x^{2^r+1} + x^2)^{2^{2r}})) \\ &= \text{Tr}_{2r}(\alpha((x^{2^{r+1}+1} + (x+b)^{2^{r+1}+1}) + (x^{2^r+2} + (x+b)^{2^r+2}) + (x^{2^r+1} + (x+b)^{2^r+1}) + b^2)) \\ &= \text{Tr}_{2r}(\alpha b^2) + \text{Tr}_{2r}(\alpha((b^{2^{r+1}+1} + b^{2^r+2})) + \text{Tr}_{2r}(\alpha b^{2^r+1}) \\ &+ \text{Tr}_{2r}(bx^{2^{r+1}} + b^{2^r}x^2) + \text{Tr}_{2r}(bx^{2^r} + b^{2^r}x) + \text{Tr}_{2r}(b^2x^{2^r} + b^{2^{r+1}}x) \\ &= \text{Tr}_{2r}(\alpha b^2) \\ &= \begin{cases} 0 & \text{if } b \in \mathbb{F}_{2^{2r}} \\ 1 & \text{if } b \in \mathbb{F}_{2^{2r}} \setminus \mathbb{F}_{2^r}. \end{cases} \end{aligned}$$

If f_α has three roots in \mathbb{F}_{2^r} (resp. one root in \mathbb{F}_{2^r} and two roots in $\mathbb{F}_{2^{2r}}$), then

$$(\alpha^{-1}x + x^2 + x^4) + (\alpha^{-1}x + x^2 + x^4)^{2^r} = (x^{2^r} + x)(x^{2^r} + x + a)(x^{2^r} + x + b)(x^{2^r} + x + c)$$

where $a, b, c \in \mathbb{F}_{2^r}$ (resp. $a \in \mathbb{F}_{2^r}$ and $b, c \in \mathbb{F}_{2^{2r}}$). If f_α has no root in \mathbb{F}_{2^r} , then $W_{3,\alpha}(4) = \mathbb{F}_{2^{2r}}$. \square

Theorem 7. *If r is odd and $x^3 + x + \alpha^{-1}$ has three roots in \mathbb{F}_{2^r} , then we have*

$$\#C_{3,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 7 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+1)/2} & \text{if } n \equiv 3, 5 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 2 \pmod{8}, \\ (2^{rn} + 1) + 2^{r(n+2)/2+1} & \text{if } n \equiv 4 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

If r is even and $x^3 + x + \alpha^{-1}$ has three roots in \mathbb{F}_{2^r} , then we have

$$\#C_{3,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 3, 5, 7 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 2 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 4 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

If r is odd and $x^3 + x + \alpha^{-1}$ has one root in \mathbb{F}_{2^r} , then we have

$$\#C_{3,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 7 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+1)/2} & \text{if } n \equiv 3, 5 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 2, 4 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

If r is even and $x^3 + x + \alpha^{-1}$ has one root in \mathbb{F}_{2^r} , then we have

$$\#C_{3,\alpha}(\mathbb{F}_{2^{rn}}) = \begin{cases} (2^{rn} + 1) + 2^{r(n+1)/2} & \text{if } n \equiv 1, 3, 5, 7 \pmod{8}, \\ (2^{rn} + 1) & \text{if } n \equiv 2, 4 \pmod{8}, \\ (2^{rn} + 1) - 2^{r(n+2)/2+1} & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

If $x^3 + x + \alpha^{-1}$ has no root in \mathbb{F}_{2^r} , the possibilities for $\#C_{3,\alpha}(\mathbb{F}_{2^{rn}})$ are given by Table 5 in the appendix.

Proof. We will prove this theorem in same manner of Theorem 5 and we will also use Corollary 2 for the case $x^3 + x + \alpha^{-1}$ has no root in \mathbb{F}_{2^r} . Note that the genus of $C_{3,\alpha}$ is 2^r for all $\alpha \in \mathbb{F}_{2^r}^\times$.

If $x^3 + x + \alpha^{-1}$ has one root or three roots in \mathbb{F}_{2^r} then $C_{3,\alpha}(\mathbb{F}_{2^{8r}})$ is maximal or minimal and if $x^3 + x + \alpha^{-1}$ has no root in \mathbb{F}_{2^r} then $C_{3,\alpha}(\mathbb{F}_{2^{24r}})$ is maximal or minimal. Since $\sqrt{2^r}$ (or $\sqrt{2^{3r}}$) cannot be written as a \mathbb{Z} -linear combination of $\{\eta \mid \eta^8 = -1\}$ by Lemma 3, if $x^3 + x + \alpha^{-1}$ has one root or three roots in \mathbb{F}_{2^r} then $C_{3,\alpha}(\mathbb{F}_{2^{8r}})$ is minimal and if $x^3 + x + \alpha^{-1}$ has zero root in \mathbb{F}_{2^r} then $C_{3,\alpha}(\mathbb{F}_{2^{24r}})$ is minimal.

If $x^3 + x + \alpha^{-1}$ has one root in \mathbb{F}_{2^r} then $\#C_{3,\alpha}(\mathbb{F}_{2^{4r}}) - (2^{4r} + 1) = 0$, if $x^3 + x + \alpha^{-1}$ has three roots in \mathbb{F}_{2^r} and r is odd (even) then $C_{3,\alpha}(\mathbb{F}_{2^{4r}})$ is maximal (minimal), if $x^3 + x + \alpha^{-1}$ has zero root in \mathbb{F}_{2^r} and r is odd (even) then $C_{3,\alpha}(\mathbb{F}_{2^{12r}})$ is maximal (minimal) by Lemma 14. \square

Lemma 15. A quadratic equation $x^2 + bx + a = 0$, $a \in \mathbb{F}_{2^r}$ and $b \in \mathbb{F}_{2^r}^\times$, has two distinct roots in \mathbb{F}_{2^r} if and only if $\text{Tr}_r(a/b^2) = 0$.

Proof. Since $b \neq 0$, we can divide the equation by b^2 . So we get $(x/b)^2 + (x/b) = a/b^2$. Hence $\text{Tr}_r(a/b^2) = 0$ if and only if $x/b \in \mathbb{F}_{2^r}$ if and only if $x \in \mathbb{F}_{2^r}$. \square

Lemma 16. Denote $f_\beta(x) = x^3 + x + \beta$, where $\beta \in \mathbb{F}_{2^r}^\times$. Let

$$M_i = \#\{f_\beta \mid f_\beta(x) \text{ has precisely } i \text{ roots in } \mathbb{F}_{2^r}\}.$$

Then we have $(M_3, M_1, M_0) = (\frac{2^{r-1}-1}{3}, 2^{r-1}-1, \frac{2^r+1}{3})$ if r is odd, $(M_3, M_1, M_0) = (\frac{2^{r-1}-2}{3}, 2^{r-1}, \frac{2^r-1}{3})$ if r is even.

Proof. The derivative of $f_\beta(x)$ is $x^2 + 1$ for all $\beta \in \mathbb{F}_{2^r}^\times$. Thus it has multiple roots if and only if 1 is a root of f_β . Moreover $x = 1$ is root of f_β if $\beta = 0$ (which is not the case) and the other root of f_0 is 0. Let $F^{\times\times} := \mathbb{F}_{2^r} \setminus \{0, 1\}$. Now assume $f_\beta(x)$ has two different roots $a, b \in F^{\times\times}$. This means $f_\beta(a) = f_\beta(b)$. It follows that $(a+b)(b^2 + ab + (a^2 + 1)) = 0$; that is, $b^2 + ab + (a^2 + 1) = 0$. By Lemma 15, $\text{Tr}_r(\frac{a^2+1}{a^2}) = \text{Tr}_r(\frac{1}{a}) + \text{Tr}_r(1)$ must be 0.

Note that $1/a$ is a permutation on the set $F^{\times\times}$ and half of \mathbb{F}_{2^r} have trace 0 and the other half of \mathbb{F}_{2^r} have trace 1. Therefore we have $M_3 = [2^{r-1} - \#\{x \in \{0, 1\} \mid \text{Tr}_r(x) = \text{Tr}_r(1)\}]/3$ and $M_1 = (2^r - 2) - 3M_3$ and $M_0 = (2^r - 1) - M_3 - M_1$. \square

Using Corollary 1 and Theorem 7, and Lemma 16, Table 6 in the appendix presents the number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_3 : y^{2^r} + y = q_3(x)$.

8 Explicit formulae for $F_q(n, 0, 0, 0)$

In this section we give an explicit formula for $F_q(n, 0, 0, 0)$. In addition we present Fourier-analysed formula for the number of points on the curves C_1 , C_2 and C_3 , and thus for $F_q(n, 0, 0, 0)$. We use the former to infer the characteristic polynomial of Frobenius of these curves and hence the decomposition of their Jacobians.

Using Lemma 6 and Eq. (10), we now combine the results contained in Tables 3, 4 and 6 to give the formula for $F_q(n, 0, 0, 0)$, presented in Table 7 in the appendix.

8.1 $C_1/\mathbb{F}_q : y^q + y = x^{q+1} + x^2$

By Fourier-analysing the formulae in Table 3, for r odd we have:

$$\#C_1(\mathbb{F}_{q^n}) = q^n + 1 - \frac{(q-1)(q-\sqrt{2q})}{4}((\sqrt{q}\omega_8)^n + (\sqrt{q}\omega_8^7)^n) - \frac{(q-1)(q+\sqrt{2q})}{4}((\sqrt{q}\omega_8^3)^n + (\sqrt{q}\omega_8^5)^n),$$

while for r even we have:

$$\#C_1(\mathbb{F}_{q^n}) = q^n + 1 - \frac{(q-1)(q-2\sqrt{q})}{4}(\sqrt{q})^n - \frac{(q-1)(q+2\sqrt{q})}{4}(-\sqrt{q})^n - \frac{(q-1)q}{4}((\sqrt{q}i)^n + (-\sqrt{q}i)^n).$$

The characteristic polynomial of Frobenius is therefore

$$P_1(X) = (X^2 - \sqrt{2q}X + q)^{\frac{(q-1)(q-\sqrt{2q})}{4}} (X^2 + \sqrt{2q}X + q)^{\frac{(q-1)(q+\sqrt{2q})}{4}}$$

if r is odd, and

$$P_1(X) = (X - \sqrt{q})^{\frac{(q-1)(q-2\sqrt{q})}{4}} (X + \sqrt{q})^{\frac{(q-1)(q+2\sqrt{q})}{4}} (X^2 + q)^{\frac{(q-1)q}{4}}$$

if r is even. These factors all arise from dimension one supersingular abelian varieties [17, Theorems 12.1 & 12.2].

8.2 $C_2/\mathbb{F}_q : y^q + y = x^{2q+1} + x^{q+2}$

By Fourier-analysing the formulae in Table 4, for r odd we have:

$$\begin{aligned} \#C_2(\mathbb{F}_{q^n}) = q^n + 1 &- \frac{(q-1)(q-\sqrt{2q})}{4}((\sqrt{q}\omega_8)^n + (\sqrt{q}\omega_8^7)^n) \\ &- \frac{(q-1)(q+\sqrt{2q})}{4}((\sqrt{q}\omega_8^3)^n + (\sqrt{q}\omega_8^5)^n) \\ &- \frac{(q-1)q}{2}((\sqrt{q}i)^n + (-\sqrt{q}i)^n). \end{aligned}$$

Let $\omega_6 = (1 + \sqrt{3}i)/2$. Then for r even we have:

$$\begin{aligned} \#C_2(\mathbb{F}_{q^n}) = q^n + 1 &- \frac{(q-1)(q-2\sqrt{q})}{12}(\sqrt{q})^n \\ &- \frac{(q-1)(q+2\sqrt{q})}{12}(-\sqrt{q})^n \\ &- \frac{(q-1)(q-\sqrt{q})}{3}((\sqrt{q}\omega_6)^n + (\sqrt{q}\omega_6^5)^n) \\ &- \frac{(q-1)(q+\sqrt{q})}{3}((\sqrt{q}\omega_6^2)^n + (\sqrt{q}\omega_6^4)^n) \\ &- \frac{(q-1)q}{4}((\sqrt{q}i)^n + (-\sqrt{q}i)^n). \end{aligned}$$

The characteristic polynomial of Frobenius is therefore

$$P_2(X) = (X^2 - \sqrt{2q}X + q)^{\frac{(q-1)(q-\sqrt{2q})}{4}} (X^2 + \sqrt{2q}X + q)^{\frac{(q-1)(q+\sqrt{2q})}{4}} (X^2 + q)^{\frac{(q-1)q}{2}}$$

if r is odd, and

$$P_2(X) = (X - \sqrt{q})^{\frac{(q-1)(q-2\sqrt{q})}{12}} (X + \sqrt{q})^{\frac{(q-1)(q+2\sqrt{q})}{12}} \cdot \\ (X^2 - \sqrt{q}X + q)^{\frac{(q-1)(q-\sqrt{q})}{3}} (X^2 + \sqrt{q}X + q)^{\frac{(q-1)(q+\sqrt{q})}{3}} (X^2 + q)^{\frac{(q-1)q}{4}}$$

if r is even. Again, these factors all arise from dimension one supersingular abelian varieties [17, Theorems 12.1 & 12.2].

8.3 $C_3/\mathbb{F}_q : y^q + y = x^{2q+1} + x^{q+2} + x^{q+1} + x^2$

By Fourier-analysing the formulae in Table 6, for r odd we have:

$$\begin{aligned} \#C_3(\mathbb{F}_{q^n}) &= q^n + 1 - \frac{(q-2)q}{8}((\sqrt{q})^n + (-\sqrt{q})^n) \\ &\quad - \frac{(q-2)q}{8}((\sqrt{q}i)^n + (-\sqrt{q}i)^n) \\ &\quad - \frac{(q+1)(q-\sqrt{2q})}{12}((\sqrt{q}\omega_{24})^n + (\sqrt{q}\omega_{24}^7)^n + (\sqrt{q}\omega_{24}^{17})^n + (\sqrt{q}\omega_{24}^{23})^n) \\ &\quad - \frac{(q+1)(q+\sqrt{2q})}{12}((\sqrt{q}\omega_{24}^5)^n + (\sqrt{q}\omega_{24}^{11})^n + (\sqrt{q}\omega_{24}^{13})^n + (\sqrt{q}\omega_{24}^{19})^n) \\ &\quad - \frac{(q-2)(5q-4\sqrt{2q})}{24}((\sqrt{q}\omega_8)^n + (\sqrt{q}\omega_8^7)^n) \\ &\quad - \frac{(q-2)(5q+4\sqrt{2q})}{24}((\sqrt{q}\omega_8^3)^n + (\sqrt{q}\omega_8^5)^n) \end{aligned}$$

For r even we have:

$$\begin{aligned} \#C_3(\mathbb{F}_{q^n}) &= q^n + 1 - \frac{(q-2\sqrt{q})(5q+2\sqrt{q}-4)}{24}(\sqrt{q})^n - \frac{(q+2\sqrt{q})(5q-2\sqrt{q}-4)}{24}(-\sqrt{q})^n \\ &\quad - \frac{q(5q-8)}{24}((\sqrt{q}i)^n + (-\sqrt{q}i)^n) \\ &\quad - \frac{q^2}{8}((\sqrt{q}\omega_8)^n + (\sqrt{q}\omega_8^7)^n) \\ &\quad - \frac{q^2}{8}((\sqrt{q}\omega_8^3)^n + (\sqrt{q}\omega_8^5)^n) \\ &\quad - \frac{(q-1)q}{12}((\sqrt{q}\omega_{12})^n + (\sqrt{q}\omega_{12}^5)^n + (\sqrt{q}\omega_{12}^7)^n + (\sqrt{q}\omega_{12}^{11})^n) \\ &\quad - \frac{(q-1)(q-2\sqrt{q})}{12}((\sqrt{q}\omega_6)^n + (\sqrt{q}\omega_6^5)^n) \\ &\quad - \frac{(q-1)(q+2\sqrt{q})}{12}((\sqrt{q}\omega_6^2)^n + (\sqrt{q}\omega_6^4)^n) \end{aligned}$$

The characteristic polynomial of Frobenius is therefore

$$P_3(X) = (X^2 - q)^{\frac{(q-2)q}{8}} (X^2 + q)^{\frac{(q-2)q}{8}} (X^2 - \sqrt{2q}X + q)^{\frac{(q-2)(5q-4\sqrt{2q})}{24}} (X^2 + \sqrt{2q}X + q)^{\frac{(q-2)(5q+4\sqrt{2q})}{24}} \\ (X^4 - \sqrt{2q}X^3 + qX^2 - q\sqrt{2q}X + q^2)^{\frac{(q+1)(q-\sqrt{2q})}{12}} \\ (X^4 + \sqrt{2q}X^3 + qX^2 + q\sqrt{2q}X + q^2)^{\frac{(q+1)(q+\sqrt{2q})}{12}}$$

if r is odd, and

$$P_3 = (X - \sqrt{q})^{\frac{(q-2\sqrt{q})(5q+2\sqrt{q}-4)}{24}} (X + \sqrt{q})^{\frac{(q+2\sqrt{q})(5q-2\sqrt{q}-4)}{24}} \\ (X^2 + q)^{\frac{q(5q-8)}{24}} (X^2 - \sqrt{q}X + q)^{\frac{q^2}{8}} (X^2 + \sqrt{q}X + q)^{\frac{q^2}{8}} \\ (X^4 - qX^2 + q^2)^{\frac{(q-1)q}{12}} (X^2 - \sqrt{q}X + q)^{\frac{(q-1)(q-2\sqrt{q})}{12}} (X^2 + \sqrt{q}X + q)^{\frac{(q-1)(q+2\sqrt{q})}{12}}$$

if r is even.

These factors all arise from dimension one and two supersingular abelian varieties [17, Theorems 12.1 & 12.2]. We remark that the three curves C_1 , C_2 , and C_3 have fairly large automorphism groups, and it may well be feasible to compute their characteristic polynomials of Frobenius by taking quotients and computing the decomposition of their Jacobian's via the Kani-Rosen theorem [9], rather than use the techniques adopted here.

Finally, combining the expressions for $\#C_1(\mathbb{F}_{q^n})$, $\#C_2(\mathbb{F}_{q^n})$ and $\#C_3(\mathbb{F}_{q^n})$ as per Lemma 6 and Eq. (10) gives the following theorem.

Theorem 8. *For r odd we have:*

$$F_q(n, 0, 0, 0) = q^{n-3} - q^{n/2-3} \left(\frac{q(q-1)(q-2)}{8} (1^n + (-1)^n) \right. \\ + \frac{q(q-1)(q+2)}{8} (i^n + (-i)^n) \\ + \frac{(q-1)(q+1)(q-\sqrt{2q})}{12} (\omega_{24}^n + \omega_{24}^{7n} + \omega_{24}^{17n} + \omega_{24}^{23n}) \\ + \frac{(q-1)(q+1)(q+\sqrt{2q})}{12} (\omega_{24}^{5n} + \omega_{24}^{11n} + \omega_{24}^{13n} + \omega_{24}^{19n}) \\ + \frac{(q-1)(q-\sqrt{2q})(5q+\sqrt{2q}+4)}{24} (\omega_8^n + \omega_8^{7n}) \\ \left. + \frac{(q-1)(q+\sqrt{2q})(5q-\sqrt{2q}+4)}{24} (\omega_8^{3n} + \omega_8^{5n}) \right).$$

For r even we have:

$$F_q(n, 0, 0, 0) = q^{n-3} - q^{n/2-3} \left(\frac{(q-1)(q-2\sqrt{q})(5q+2\sqrt{q}+4)}{24} 1^n \right. \\ + \frac{(q-1)(q+2\sqrt{q})(5q-2\sqrt{q}+4)}{24} (-1)^n \\ + \frac{q(q-1)(5q+4)}{24} (i^n + (-i)^n) \\ + \frac{q^2(q-1)}{8} (\omega_8^n + \omega_8^{7n} + \omega_8^{3n} + \omega_8^{5n}) \\ + \frac{q(q-1)^2}{12} (\omega_{12}^n + \omega_{12}^{5n} + \omega_{12}^{7n} + \omega_{12}^{11n}) \\ + \frac{(q-1)(q-\sqrt{q})(q-\sqrt{q}+2)}{12} (\omega_6^n + \omega_6^{5n}) \\ \left. + \frac{(q-1)(q+\sqrt{q})(q+\sqrt{q}+2)}{12} (\omega_6^{2n} + \omega_6^{4n}) \right).$$

9 Concluding remarks

By Fourier-analysing known formulae which count the number of elements of \mathbb{F}_{2^n} for which the first three coefficients of the characteristic polynomial with respect to \mathbb{F}_2 are prescribed, we developed a

new much simpler curve-based approach to deriving them, in the trace zero cases for all n and the trace one cases for odd n . This approach was used to count the number of irreducible polynomials in $\mathbb{F}_{2^r}[x]$ for which the first three coefficients are zero. Based on the \mathbb{F}_2 base field case and our result for $f_q(n, 0, 0, 0)$, we conjecture that for $q = 2^r$, $n \geq 3$ and $t_1, t_2, t_3 \in \mathbb{F}_q$, the formulae for $f_q(n, t_1, t_2, t_3)$ all have period 24.

For odd n it is possible to compute all of the trace zero cases with our approach, but for reasons of space we only sketch the method here. First note that although one can express $N(0, 0)$ in terms of $Z(q_1)$, $Z(q_2)$ and $Z(q_1 + q_2)$ as per Lemma 6, it is not possible to express all $N(t_2, t_3)$ in terms of these. For example, for any $t_2 \in \mathbb{F}_q^\times$ one has $Z(q_1) = Z(t_2 q_1) = \sum_{t_3 \in \mathbb{F}_q} N(0, t_3)$, so one does not get a linear system of full rank. However, this issue can be obviated by changing the definition of q_1 and q_2 to:

$$\begin{aligned} q_1 : \mathbb{F}_{q^n} &\rightarrow \mathbb{F}_q : x \mapsto T_1(x^{q+1} + x^2) + t_2, \\ q_2 : \mathbb{F}_{q^n} &\rightarrow \mathbb{F}_q : x \mapsto T_1(x^{2q+1} + x^{q+2}) + t_3. \end{aligned}$$

Then, for odd n one has $T_1(t_2) = t_2$ and $T_1(t_3) = t_3$, and so computing the number of points on the supersingular curves $C' : y^q + y = x^{q+1} + x^2 + t_2$, $C'_\alpha : y^q + y = x^{2q+1} + x^{q+2} + t_3 + \alpha(x^{q+1} + x^2 + t_2)$ for $\alpha \in \mathbb{F}_q$ will give expressions for $F_q(n, 0, t_2, t_3)$.

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Appendix: point and trace counts

Table 3. The number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_1 : y^{2^r} + y = q_1(x)$

$n \pmod{8}$	r odd	r even
0	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
1	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
2	$2^{rn} + 1$	$2^{rn} + 1$
3	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
4	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
5	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
6	$2^{rn} + 1$	$2^{rn} + 1$
7	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$

Table 4. The number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_2 : y^{2^r} + y = q_2(x)$

$n \pmod{24}$	r odd	r even
0	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$
1	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
2	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$
3	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$
4	$2^{rn} + 1$	$2^{rn} + 1$
5	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
6	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
7	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
8	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$	$2^{rn} + 1$
9	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$
10	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$
11	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
12	$2^{rn} + 1$	$2^{rn} + 1 - (2^r - 1)2^{r(n+2)/2+1}$
13	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
14	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$
15	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$
16	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$	$2^{rn} + 1$
17	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
18	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
19	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
20	$2^{rn} + 1$	$2^{rn} + 1$
21	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$
22	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+2)/2}$
23	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$

Table 5. The number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_3 : y^{2^r} + y = q_3(x)$ when $x^3 + x + \alpha^{-1}$ has no root in \mathbb{F}_{2^r}

$n \pmod{24}$	r odd	r even
0	$(2^{rn} + 1) - 2^{r(n+2)/2+1}$	$(2^{rn} + 1) - 2^{r(n+2)/2+1}$
1	$(2^{rn} + 1) + 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
2	$2^{rn} + 1$	$2^{rn} + 1$
3	$(2^{rn} + 1) + 2^{r(n+1)/2+1}$	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$
4	$(2^{rn} + 1) - 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
5	$(2^{rn} + 1) - 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
6	$2^{rn} + 1$	$2^{rn} + 1$
7	$(2^{rn} + 1) + 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
8	$(2^{rn} + 1) + 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
9	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$
10	$2^{rn} + 1$	$2^{rn} + 1$
11	$(2^{rn} + 1) - 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
12	$(2^{rn} + 1) + 2^{r(n+2)/2+1}$	$(2^{rn} + 1) - 2^{r(n+2)/2+1}$
13	$(2^{rn} + 1) - 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
14	$2^{rn} + 1$	$2^{rn} + 1$
15	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$
16	$(2^{rn} + 1) + 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
17	$(2^{rn} + 1) + 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
18	$2^{rn} + 1$	$2^{rn} + 1$
19	$(2^{rn} + 1) - 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$
20	$(2^{rn} + 1) - 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
21	$(2^{rn} + 1) + 2^{r(n+1)/2+1}$	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$
22	$2^{rn} + 1$	$2^{rn} + 1$
23	$(2^{rn} + 1) + 2^{r(n+1)/2}$	$(2^{rn} + 1) + 2^{r(n+1)/2}$

Table 6. The number of $\mathbb{F}_{2^{rn}}$ -rational points on $C_3 : y^{2^r} + y = q_3(x)$

$n \pmod{24}$	r odd	r even
0	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2+1}$
1	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
2	$2^{rn} + 1$	$2^{rn} + 1$
3	$(2^{rn} + 1) + 2^{r(n+1)/2+1}$	$2^{rn} + 1$
4	$(2^{rn} + 1) - 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
5	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
6	$2^{rn} + 1$	$2^{rn} + 1$
7	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
8	$(2^{rn} + 1) - (2^r - 3)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
9	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$	$2^{rn} + 1$
10	$2^{rn} + 1$	$2^{rn} + 1$
11	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
12	$(2^{rn} + 1) + 2^{r(n+4)/2}$	$2^{rn} + 1 - (2^r - 2)2^{r(n+2)/2}$
13	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
14	$2^{rn} + 1$	$2^{rn} + 1$
15	$(2^{rn} + 1) - 2^{r(n+1)/2+1}$	$2^{rn} + 1$
16	$(2^{rn} + 1) - (2^r - 3)2^{r(n+2)/2}$	$(2^{rn} + 1) - (2^r - 1)2^{r(n+2)/2}$
17	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
18	$2^{rn} + 1$	$2^{rn} + 1$
19	$(2^{rn} + 1) - (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$
20	$(2^{rn} + 1) - 2^{r(n+2)/2}$	$(2^{rn} + 1) + 2^{r(n+2)/2}$
21	$(2^{rn} + 1) + 2^{r(n+1)/2+1}$	$2^{rn} + 1$
22	$2^{rn} + 1$	$2^{rn} + 1$
23	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$	$(2^{rn} + 1) + (2^r - 1)2^{r(n+1)/2}$

Table 7. $F_q(n, 0, 0, 0)$

$n \pmod{24}$	r odd	r even
0	$q^{n-3} - (q-1)(2q+1)q^{n/2-2}$	$q^{n-3} - (q-1)(2q+1)q^{n/2-2}$
1	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
2	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} + (q-1)q^{n/2-2}$
3	q^{n-3}	q^{n-3}
4	q^{n-3}	q^{n-3}
5	$q^{n-3} - (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
6	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} - (q-1)q^{n/2-2}$
7	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
8	$q^{n-3} - (q-1)q^{n/2-1}$	$q^{n-3} - (q-1)q^{n/2-1}$
9	q^{n-3}	q^{n-3}
10	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} + (q-1)q^{n/2-2}$
11	$q^{n-3} - (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
12	$q^{n-3} + (q^2-1)q^{n/2-2}$	$q^{n-3} - (q^2-1)q^{n/2-2}$
13	$q^{n-3} - (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
14	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} + (q-1)q^{n/2-2}$
15	q^{n-3}	q^{n-3}
16	$q^{n-3} - (q-1)q^{n/2-1}$	$q^{n-3} - (q-1)q^{n/2-1}$
17	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
18	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} - (q-1)q^{n/2-2}$
19	$q^{n-3} - (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$
20	q^{n-3}	q^{n-3}
21	q^{n-3}	q^{n-3}
22	$q^{n-3} + (q-1)q^{n/2-2}$	$q^{n-3} + (q-1)q^{n/2-2}$
23	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$	$q^{n-3} + (q^2-1)q^{(n-1)/2-2}$